UNIT 7 INTEGRATION AND APPLICATIONS IN ECONOMIC DYNAMICS

Structure

7.0 OBJECTIVES

After going through this unit, you should be able to:

- identify the dynamics problems in economics; and
- use the mathematical tools of differential equation to solve problems related to economic theory.

7.1 INTRODUCTION

Events that change over time are put under the purview of dynamic analysis. In this unit, we introduce a framework for dealing with dynamic economic problems by introducing time explicitly into these. For that purpose, let us start with the mathematical techniques of integral calculus and differential equations.

7.2 DYNAMICS AND INTEGRATION

In a dynamic economics model, the basic objective is the identification of the time path of the variable on the basis of its rate of change. For example, national income y of a country changes overtime. To see the rate of change we need to see its change with respect to time and to find the time path followed by y. Thus, if we know the derivative $\frac{dy}{dt}$, it will be possible to get onto the function like $y = y(t)$ through the technique of integration which happens to be opposite of the process of differentiation. We will return to this process after a while.

7.3 THE TOOLS OF DYNAMICS

The dynamic exercises are usually carried on with the aid of differential and difference equations. By taking 'time' as the independent variable, an attempt is made to derive their solutions. The simple integrals, both indefinite and definite, help in many contexts to define some important concepts of economic dynamics. The discussion that follows begins by introducing the notion of the integral. Then we move on to discuss the technique of differential equations. From now on, all functions in this unit will be assumed to be continuous and real valued.

7.3.1 The Indefinite Integral and its Economic Applications

As we have pointed out above, basically indefinite integral is reverse differentiation. Recall that differential calculus gives the rate of change (derivative) of a given function. Indefinite integration reverses such a process and finds the unknown function whose rate of change (derivative) is given. If the function f(x) is written symbolically as

$$\int f(x)dx$$

which is read as "the integral of *f* (x) with respect to x" and *f* (x) is the 'integrand'. We can write

$$\int f(x)dx = F(x)$$

provided f(x) is the derivative of $F(x)$ i.e. f(x) = $\frac{d}{dx}$F(x). It is not difficult to see that the function F(x) is not unique, because if F(x) contains the derivative *f*(x), then so has F(x) + c where c is any arbitrary constant. For that reason the indefinite integral is always written with an arbitrary constant, called 'the constant of integration'. As functions differ only by an additive constant, the derivative remains the same.

Examples:

1) $\int (x^3 + 3)dx = \frac{1}{4}x^4 + 3x + c$

2) $\int e^x dx = e^x + c$

3) $\int \frac{1}{x} dx = \log(x) + c$

Through the above examples we may verify that in each case the derivative of the right hand side equals the corresponding integrand on the left-hand side. Thus we see that

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

We now state two useful rules of integration.

Rule 1: The integral of a constant time a function equals the constant times the integral of the function.

$$\int kf(x)dx = k\int f(x)dx$$

Example: $\int 2x^2dx = 2\int x^2dx = 2\left(\frac{x^3}{3}+c\right)$

If we recall that the derivative of a constant time a function is the constant times the derivative of the function, it will be easier for us to appreciate above rule. If k = –1, we have the result

$$\int [-f(x)]dx = -\int f(x)dx.$$

Rule 2: The integral of a sum of functions equals the sum of the integrals of the functions, viz.,

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx.$$

Example: $\int (5e^x - x^{-2} + \frac{3}{x})dx = \int (5e^x)dx - \int (x^{-2})dx + \int \frac{3}{x}dx$

$$= 5\int e^x dx - \int x^{-2}dx + 3\int \frac{1}{x}dx$$

$$= \left(5e^x + c_1\right) - \left(\frac{x^{-1}}{-1} + c_2\right) + \left(3\log x + c_3\right)$$

$$= 5e^x + \frac{1}{x} + 3\log x + c$$

The rule is directly related to that of the derivative of a sum of function is the sum of the derivatives of the functions.

Some Useful Formulae

We state without proof some useful formulae for integration.

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

2) $\int x^{-1}dx = \int \frac{dx}{x} = \log x + c$ for any x > 0

3) $\int e^{mx}dx = \frac{e^{mx}}{m} + c$ for any m.

4) $\int a^{mx}dx = \frac{a^{mx}}{m\log a} + c$

5) $\int \cos ax dx = \frac{\sin ax}{a} + c$

6) $\int \sin ax dx = -\frac{\cos ax}{a} + c$

7) $\int [K_1 f(x) + K_2 g(x) dx] = K_1 \int f(x) dx + K_2 g(x) dx + c$ for $K_1, K_2 \geq 0$

Determination of the Constant of Integration using Initial or Boundary Conditions

We have seen above that the indefinite integral is assigned an arbitrary constant. Its value can be precisely determined if we know that the integral of the function y = *f* (x) + c obeys some prescribed **initial condition** ($y = y_0$ when x = 0) or more generally **boundary condition** ($y = y_0$ when $x = x_0$). To understand the underlying idea let us take the following example:

$$\int (x^3 + x + 1) dx = \left(\frac{x^4}{4} + c_1\right) + \left(\frac{x^2}{2} + c_2\right) + (x + c_3)$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + x + c$$

Here y = *f*(x) + c = $\frac{x^4}{4} + \frac{x^2}{2} + x + c$. Suppose, we are given the initial condition, y = 20 when x = 0. Then

$y_0 = f(0) + c = c = 20$

This fixes the integral as the unique function y = $\frac{x^4}{4} + \frac{x^2}{2} + x + 20$. We will return to the initial condition again in Section 7.4.

Some Computational Methods

The standard procedure of integration given above is sometimes inadequate for computational purposes. In such a situation, it becomes necessary to attempt certain kinds of manipulation before the function becomes amenable to integration. We have a few standard results to help initiate the process of integration. However, remember that these are not all that can be used and there are no routine manipulations that can be prescribed. Practice opens up the channels of finding a solution to the integral.

Method of Substitution

If F(x) = $\int f(x) dx$, the indefinite integral can be obtained by resorting to transformation. If we take x = g(y), then

$$\int f(x) dx = \int [g(y)] g'(y) dy.$$

See that $x = g(y) \Rightarrow dx = g'(y) dy$.

If $\phi(y) = \int f[g(y)] g'(y) dy$, then

$F(x) = \phi[g^{-1}$ (.) is the inverse of g(.)].

Example: i) Solve $F(x) = \int (1+5x)^{\frac{1}{2}} dx$.

Let y = 1 + 5x

dy = 5dx, or dx = $\frac{1}{5}$dy

$\therefore F(x) = \frac{1}{5}\int y^{\frac{1}{2}} dy = \frac{1}{5}\cdot\frac{2}{3} y^{\frac{3}{2}} + c$

= $\frac{2}{15}\cdot y^{\frac{3}{2}} + c = \frac{2}{15}(1+5x)^{\frac{3}{2}} + c$

ii) Solve $F(x) = \int \frac{dx}{(3x-1)^2}$

Let y = 3x – 1

So, dy = 3 dx, or, dx = $\frac{1}{3}$dy.

$\therefore F(x) = \frac{1}{3}\int \frac{dy}{y^2} = \frac{1}{3}\left(-\frac{1}{y}\right) + c = \frac{1}{3(3x+1)} + c$

Integration by Parts

Let us return to differentiation of product of two functions u = *f*(x) and v = *f*(x) which gives,

d(uv) = $u\frac{dv}{dx} + v\frac{du}{dx}$.

From this, we can obtain

$\int (uv)' dx = \int uv' dx + \int vu' dx$

$\Theta \int (uv)' dx = uv$

$\therefore uv' dx = uv - \int vu' dx$

Example: Solve F(x) = $\int xe^{-x} dx$

Let u(x) = x, v(x) = $-e^{-x}$

So that u′(x) = 1, v′(x) = e^{-x}

$\therefore f(x) = \int u.v' dx$

$= uv - \int vu' dx$

$$= -xe^{-x} + \int e^{-x} dx$$

$$= -xe^{-x} - e^{-x} + c = -e^{-x}(1+x) + c$$

Economic Applications of Indefinite Integration

Consider the following two examples as a part of your exercise to apply the tool of indefinite integration to often cited problems of economics.

a) **Investment and the Stock of Capital**

Let net investment I is the rate of change of the stock of capital K. If time is treated as a continuous variable, we can express this as

$$I(t) = \frac{dK(t)}{dt}.$$

Thus, if the rate of investment I(t) is known, the capital stock K(t) can be estimated through the formula,

$$K(t) = \int I(t)dt$$

Example: The rate of net investment is given by $I(t) = 12t^{\frac{1}{3}}$ and the initial stock of capital at t = 0 is 25 units. Find the equation for the stock of capital.

$$K(t) = \int 12t^{\frac{1}{3}} dt = 12\left(\frac{3}{4}\right)t^{\frac{4}{3}} + c$$

$$= 9t^{\frac{4}{3}} + c$$

As K(0) = c = 25 given,

$$K(t) = 9t^{\frac{4}{3}} + 25$$

b) **Obtaining the Total from the Margin**

Integration helps us recover the total function from the marginal function if the concerned variable varies continuously. Thus, it will be possible to derive the total functions such as cost, revenue, production and saving from their marginal functions. We will examine a simple application to see the procedure involved.

Example: If the marginal revenue function of a firm in the production of output is MR = 40 – $10q^2$ where q is the level of output and total revenue is 120 at 3 units of output, find the total revenue function.

Since MR = $\frac{dTR}{dq}$, we can write

$$TR = \int MR dq$$

$$= \int (40 - 10q^2)\, dq$$

$$= 40q - \frac{10}{3}q^3 + c$$

At q = 3, TR = 30 + c = 100 given. So c = 90. The required total revenue function is

TR(q)= $40q - \frac{10}{3}q^3 + 90$.

7.3.2 The Definite Integral and its Economic Applications

The definite integral of the function *f*(x) over the interval [a, b] is expressed symbolically as $\int_a^b f(x)dx$, read as "the integral of *f* with respect to x from a to b".

The smaller number a is termed the **lower limit** and b, the **upper limit**, of integration. Geometrically, this definite integral denotes the area under the curve representing *f*(x) between the points x = a and x = b.

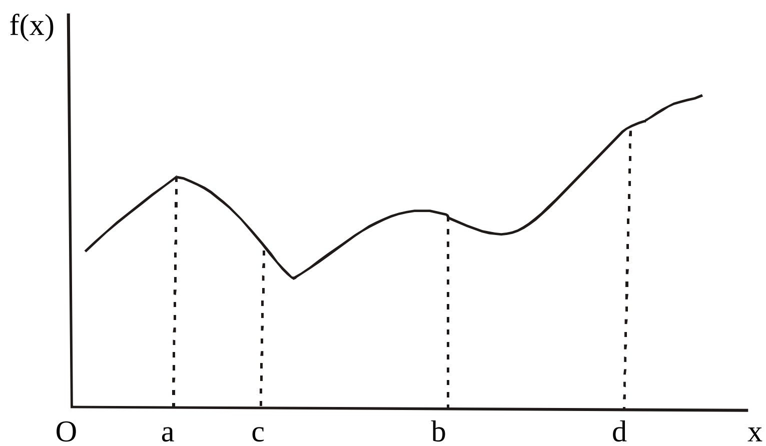

Fig. 7.1

It should be noted that the indefinite integral *f*(x) dx is a **function** of x, whereas the definite integral $\int_a^b f(x)dx$ is a **number**. The numerical value of the definite integral depends on the two limits of integral also changes. This is clear from Figure 7.1 where if we change the interval (a, b) to (c, d) the value of the area under the curve will, in general, change.

Another feature of the definite integral is that its value does not depend on the particular symbol chosen to represent the independent variable so long as the form of the function is not changed. That is,

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du = \text{etc.}$$

The following theorem establishes the connection between indefinite and definite integration and supplies the method for evaluating definite integrals.

The Fundamental Theorem of Calculus

If $\int_a^b f(x)dx = f(x) + c$, then

$$\int_a^b f(x)dx = f(b) - f(a).$$

Examples: 1) To evaluate $\int_1^5 x^2 dx$.

$$\int_1^5 x^2 dx = x^3 + c$$

So, $\int_1^5 x^2 dx = 5^3 - 1^3 = 124.$

2) To evaluate $\int_{-1}^{1} (ax^2 + bx + c)dx.$

$$\int (ax^2 + bx + c)dx = a\frac{1}{3}x^3 + b\frac{1}{2}x^2 + cx + c'.$$

So, $\int_{-1}^{1} (ax^2 + bx + c)dx = \left(a\frac{1^3}{3} + b\frac{1^2}{2} + c.1 \right) - \left(a\frac{(-1)^3}{3} + b\frac{(-1)^2}{2} + c.(-1) \right)$

$$= (\frac{1}{3}a + \frac{1}{2}b + c) - (-\frac{1}{3}a + \frac{1}{2}b - c)$$
$$= (\frac{1}{3}a - \frac{1}{2}b + c)$$
$$= \frac{2}{3}a + 2c$$
$$= 2(\frac{1}{3}a + 1)$$

Definite integrals are subject to certain rules of operation.

Rule 1: If the two limits are equal, the value of the integral is zero.

$$\int_a^b f(x)dx = 0.$$

Rule 2: Reversing the limits of integration changes the sign of the integral.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Rule 3: The definite integral can be expressed as the sum of subintegrals.

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

where b is a point within the interval (a, c).

We now discuss briefly one special type of definite integral, the **improper integral**. When one of the limits of integration is + ∞ or (–∞) a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits according to the following rules:

i) $\int_a^\infty f(x)dx = \underset{b\to\infty}{Lt} \int_a^b f(x)dx$.

ii) $\int_{-\infty}^b f(x)dx = \underset{a\to\infty}{Lt} \int_a^b f(x)dx$

Example:

Evaluate $\int_1^\infty \frac{dx}{x^2}$

Since $\int_1^b \frac{dx}{x^2} = -\frac{1}{b} + 1$, the desired integral is

$$\underset{b\to\infty}{Lt} \int_1^b \frac{dx}{x^2} = \underset{b\to\infty}{Lt}\left(-\frac{1}{b}\right) = 1.$$

Economic Applications of the Definite Integral

a) **Consumer's Surplus**

Consumer's surplus (CS) measures the net benefit that a consumer enjoys from the purchase of a particular commodity in the market. To measure CS, we take (i) the demand function of a consumer P = *f*(q) representing the highest price a consumer is willing to pay (her 'demand price') for any specified quantity, (ii) the actual price paid for the quantity purchased and (iii) get the difference between (i) and (ii). In the figure below, a consumer is willing to pay a price of p_1 per unit for q_1 units, p_2 per unit for q_2 units, and so on. Suppose the market price is $\overline{p}$. At this prices she purchases $\overline{q}$ units and her actual expenditure is $\overline{pq}$, represented by the rectangle $O\overline{p}E\overline{q}$. Her total willingness to pay for $\overline{q}$ is obtained as the sum of her demand prices for all the units from 0 to $\overline{q}$.

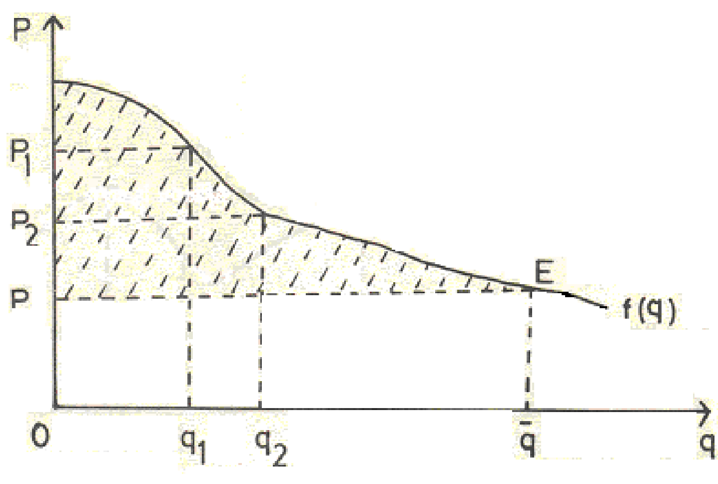

Fig. 7.2

Mathematically, this is the definite integral of the demand function up to $\overline{q}$, or the area under the demand curve up to $\overline{q}$. The excess of this total willingness to pay in units of money over her actual expenditure is her Consumer's surplus.

$$CS = \int_0^{\bar{q}} f(q)dq - \bar{p}\bar{q}.$$

It is represented by the crossed area in the diagram.

Example: Suppose the demand function of a consumer is given by p = 80 – q. If the price offered is p = 60, find the consumer surplus.

For p = 60, we get q = 20 from the demand equation. Actual expenditure pq = 1200.

Now $CS = \int_0^{20} (80 - q)dq - pq$

$= 1400 - 1200 = 200$.

Thus the consumer's surplus is Rs.200.

b) **Capital Accumulation Over a Specified Period**

Since $\int I(t)dt = K(t) + c$, we may use the definite integral

$\int_a^b I(t)dt = K(b) - K(a)$ to find the total capital accumulation during the time interval [a, b].

Example: Given the rate of net investment I(t) = $9t^{1/2}$, find the level of capital formation in (i) 16 years and (ii) between the 4^{th} and the 8^{th} years.

i) K = $\int_0^{16} 9t^{1/2}dt = 6(16)^{3/2} - 0 = 384$

ii) K = $\int_4^{8} 9t^{1/2}dt = 6(8)^{3/2} - 6(4)^{3/2} = 135.76 - 48 = 87.76.$

c) **Present Value or Discounted Value Under Continuous Compounding of Interest**

A basic concept in capital theory is the present or discounted or capital value of a specified sum of money that will be available at a future date. If the annual rate of interest is 100r percent, then the present value Y of Rs. x available next year is Y = $\frac{x}{1+r}$, because Rs. $\left(\frac{x}{1+r}\right)$ now will become Rs.x after one year at the stipulated annual rate of interest of 100r per cent. Similarly, the present value of Rs.x available t years hence is

$$Y = \frac{x}{(1+r)^t}$$

If interest is compounded n times a year at 100r per cent per year then the present value is

$$Y = \frac{x}{\left(1+\frac{r}{n}\right)^{nt}} = x\left(1+\frac{r}{n}\right)^{nt} \qquad \ldots (1)$$

If interest is compounded continuously, then n $\rightarrow \infty$ and the continuous counterpart of (1) becomes

$Y = x\, e^{-rt}$

using the result: $\underset{n\to\infty}{Lt}\left(1+\frac{k}{n}\right)^{nx} = e^{kx}$.

Now consider a project that yields an income x(t) at future period t for t = 1, 2, …, T. That is, the income stream associated with the project for T years is : x(1), x(2), …, x(T). The present or discounted value of this income stream at annual compounded is:

$$Y = \sum_{t=1}^{T} \frac{x(t)}{(1+r)^t} \qquad \ldots (2)$$

When income flows continuously at the rate of x(t) per period up to period T and interest is compounded continuously the expression for present value becomes

$$Y = \int_0^T x(t)e^{-rt}dt \qquad \ldots (3)$$

Note that the magnitude of present value depends on the size of the income stream, the number of years it flows (the time horizon) and the rate of interest (the discount factor).

You should keep in mind the distinction between the present value of the **sum** x(T) available T periods hence and the present value of the **stream of income** x(t) per period up to period T. In the figure the former is the ordinate at t = T, whereas the latter is the shaded area under the curve upto t = T.

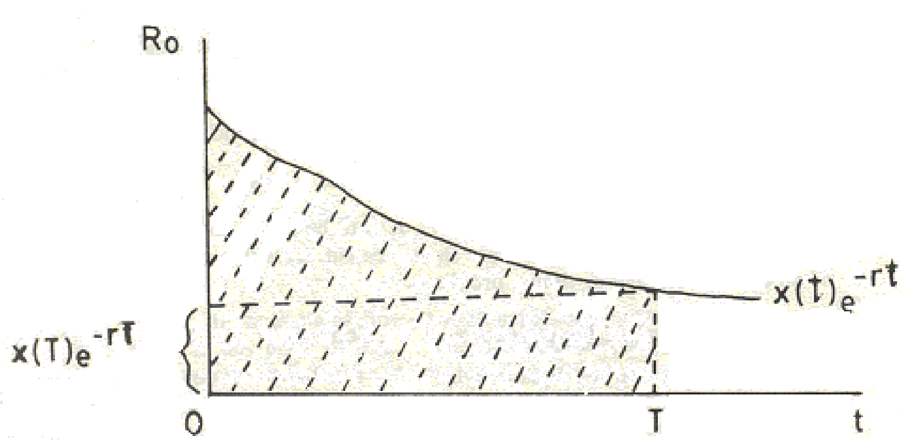

Fig. 7.3

A particular case of interest is the valuation of an asset (a bond or a piece of land) yielding a fixed income Rs. R for ever. The market value Y of such an asset is the present value of the perpetual yield.

$$Y = \int_0^\infty \text{Re}^{-rt}\, dt = R\int_0^\infty e^{-rt} dt$$

Remembering the rule for evaluating improper integrals.

$$\int_0^\infty e^{-rt} dt = \underset{b\to\infty}{Lt}\int_0^b e^{-rt} dt = \underset{b\to\infty}{Lt}\left(-\frac{1}{r}e^{-rb} + \frac{1}{r}\right) = \frac{1}{r}.$$

Hence, the market value is:

$$Y = \frac{R}{r}.$$

To illustrate further, the use of the concept of present value, we consider the following more complex problem of **optimal timing**.

The value of timber planted on a plot of land grows over time according to the function $V(t) = 2^{\sqrt{t}}$. Assuming zero cost of maintenance and a discount factor of r, find the optimal time to cut the timber for sale.

Since cost of production (upkeep) is zero, profit maximisation here is equivalent to the maximisation of sales revenue V. Due to the interest factor, r, different V values, however, are not comparable because they accrue at different points of time. The solution involves discounting each V value to its present value (the value at t = 0). The process of discounting puts them on comparable footing.

Assuming continuous compounding, the present value R(t) can be written as

$$R(t) = V(t)e^{-rt} = 2^{\sqrt{t}}\, e^{-rt}.$$

The optimal time of cutting is the value of t that maximises R(t). Since $f(x)$ and log $f(x)$ attain their maximum at the same value of x, the problem can equivalently be restated as finding the value of t that maximises log R(t).

$$\ln R(t) = \sqrt{t}\,\log 2 - rt$$

Differentiating with respect to t and setting the derivative equal to zero, we get

$$\frac{1}{R} - \frac{dR}{dt} = \left(\frac{\ln 2}{2\sqrt{t}} - r\right) = 0$$

$$\text{or, } \frac{dR}{dt} = R\left(\frac{\ln 2}{2\sqrt{t}} - r\right) = 0 (R \neq 0)$$

$$\text{or, } \sqrt{t} = \frac{\ln 2}{2r}, \quad \text{since } R \neq 0.$$

or, t = $\left(\dfrac{\ln 2}{2r}\right)^2$

We leave it to you to check that at this value of t the second order condition for maximisation $\dfrac{d^2R}{dt^2} < 0$ is also satisfied. Thus, the expression for the optimum time for cutting the timber is $(\log 2/2r)^2$. It is to be noted that the higher the rate of discount r, the sooner the timber should be cut. This is a general characteristic of all optimal storage or timing problems.

7.4 DIFFERENTIAL EQUATIONS AND ITS ECONOMIC APPLICATIONS

We deal with many economic models which have temporal dimensions involving relationships between the values of variables at a given point of time and the changes in these values over time. As an expample we may consider a model of economic growth that often postulates a functional relationship between the change in the capital stock and the value of output. When time is modelled as a continuous variable, differential equations are formulated by involving the derivatives (or differentials) of unknown functions.

7.4.1 Solving Differential Equations

Solving a differential equation means finding a function that satisfies that equation.

Let us start with some basic ideas behind these equations. If y = *f*(x) is a function for which derivatives of adequate order exist, then $\dfrac{dy}{dx} = f'(x)$. Suppose that we know *f* ′(x) and would like to go back to the function y. Therefore, we try to solve the problem.

dy = f'(x)dx

$\Rightarrow$ y = $\int f'(cx)dx$.

Through differential equations, we attempt to solve the problems, which are related to change over time, i.e., dynamic variables. For example, suppose that a hypothetical economy's income (y) is related to time (x). It is given in functional form: $y(x) = 2x^{1/2}$. If the income changes over time, we find the rate of change as $\dfrac{dy}{dx} = x^{-\frac{1}{2}}$. Let us work to find the time path of the income change, so that we write y = y(x). The derivative of this function, however, will be same as that of y = y(x) + c, where c is any arbitrary constant. In such a situation, we cannot determine a unique time path of the income change. It is necessary, therefore, to work out a definite value of c. Additional information required for that purpose is to have the *initial condition*. If we know the initial income of the economy, say, y(0), i.e., value of y at x = 0, then the value of the constant c can be determined.

Thus, from $y(x) = 2x^{1/2} + c$, when x = 0, we get y(0) = $2(0)^{1/2}$ +c = c.

See that constant c is no longer arbitrary as if y(0) = 10,000, c = 10,000 and

y(x) =$2x^{1/2}$ + 10,000. More generally, for any given initial income, y(0), the time path will be

y(x) =$2x^{1/2}$ + y(0).

Note that the income example, its dynamic form, consists of the sum of initial condition and another term with time variables.

Remember a general principle on the initial value problem: The differential equation that involves only the first derivative, has a unique solution if it has one initial condition. In addition, the differential equation that involves only the first and second derivatives, has a unique solution if it has two initial conditions.

Differential Equation: Equilibrium and Stability

In a difference equation, if the initial value has a solution that is a constant function and hence independent of *t*, then the value of the constant is called an **equilibrium state** or **stationary state** of the differential equation.

Example:

Consider the differential equation

$$y'(t) + y(t) = 2.$$

The general solution of this equation, as we shall below, is

$$y(t) = Ce^{-t} + 2.$$

Thus for the initial condition y(0) = 2, the solution of the problem is y(*t*) = 2 for all *t*. Thus the equilibrium state of the system is 2.

The **order** of a differential equation is the order of the highest derivative appearing in the equation. Its **degree** is the highest power to which the highest order derivative is raised. A differential equation is **linear** if the dependent variable and derivatives are raised to the first power only and no product term y $\frac{dy}{dx}$ occurs.

Examples: In all the examples that follow the unknown function is y = *f*(x).

1) $\frac{dy}{dx} = 4x - 9$ First order, first degree

2) $\left(\frac{dy}{dx}\right)^4 - x^2 = 0$ First order, fourth degree

3) $\frac{d^2y}{dx^2} - 2y = 0$ Second order, first degree

4) $\frac{d^4y}{dx^4}+x^3\frac{d^3y}{dx^3}-\log x\frac{d^2y}{dx^2}+e^x\frac{dy}{dx}+x^2y+10=0$ Four order, first degree

5) $\left(\frac{d^3y}{dx^3}\right)^4+\left(\frac{d^2y}{dx^2}\right)^7=8+2y$ Third order, fourth degree.

Equation (1), (3) and (4) are linear since they are all of the first degree.

Please note that depending upon the complexity of the equations that we use in course of the following discussion, the notations adopted will be in the form of either $\frac{dy}{dx} or, f'(x)$.

First Order Differential Equation

Solution to first order differential equation in specific instances can be worked out with (1) separation of variables. Consider the equation,

$$\frac{dy}{dx}=f'(x)$$

or, dy = f'(x)dx

If we integrate both the sides,

$\int dy=\int f'(x)dx$, so that variables x and y are separated, it becomes easy for applying appropriate technique of integration.

Example: $\frac{dy}{dx}=x^2$

or, dy = x^2dx

So, $\int dy=\int x^2dx$

or, y = $\frac{x^3}{3}+c$

Exact equations

From $\frac{dy}{dx}=f'(x)$, we get

y = $\int f'(x)dx+c$ which can be written as g(x) + c.

Remember that the addition of a constant term, ‘c’ to the function does not affect its derivative. However, it shifts the function parallelly. Depending upon the different values acquired by the constant term, we get a family of curves for the function. Take, for example, the above solution, y = $\frac{x^3}{3}+c$ with c= 0. Then $\frac{dy}{dx}=x^2$, or, $dy = x^2dx$

or, $dy - x^2dx = 0$

Note that '0' in the right side of this solution is due to a choice like c = 0. So we write

$dy - x^2dx = c$, (where c = constant)

Generalising the above result, we can write

$u(x, y) = c$.

So, $du = u_xd_x + u_yd_y = 0$

If $u = \frac{-x^3}{3} + y$, then $du = -x^2dx + dy = 0$

or, $\frac{dy}{dx} = x^2$

Since $u = \frac{-x^3}{3} + y$, $\frac{\partial u}{\partial x} = -x^2$ and $\frac{\partial u}{\partial y} = 1$.

Therefore, $dx = \frac{\partial u}{\partial x}$ and $dy = \frac{\partial u}{\partial y}$. From these we write the generalised form of differential equation as $P(x, y)\, dx + Q(x, y)\, dy = 0$.

If we can find a function g(x, y) such that $P(x, y) = \frac{\partial g}{\partial x}$ and $Q(x, y) = \frac{\partial g}{\partial y}$, we write

$$d\,[g(x, y)] = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial x}\, dy$$

$$= P(x, y)\, dx + Q(x, y)\, dy.$$

Then g(x, y) = c are integral curves of the above differential equation. This class of differential equations is called exact differential equations.

Example:

For $xdx + ydy = 0$, set $g = \frac{1}{2}(x^2+y^2)$. Then $g_x = xdx$ and $g_y = ydy$. The solutions are $x^2+y^2 = c$, or circles.

To determine if a differential equation is exact or not, check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Example:

From, $(3x^2+y^2)dx + 2xydy = 0$, we get

$P(x, y) = 3x^2+y^2$

Q(x, y) = 2xy, so that

$P_y = 2y = Q_x$. The equation, therefore, is exact.

Exercise:

Solve the exact differential equation

$2yxdy + y^2dx = 0$

In this equation

P(x, y) = 2yx

$Q(x, y) = y^2$

Now, $F(y, x) = \int 2yxdy + \phi(x) = y^2 x + \phi(x)$

$$\frac{dF}{dx} = y^2 + \phi'(x)$$

As $Q = \frac{dF}{dx}$, we equate $Q(x, y) = y^2$ and $\frac{dF}{dx} = y^2 + \phi'(x)$ to get $\phi'(x) = 0$

$\phi'(x) = \int \phi'(x)dx = \int 0dx = k$, which give the specific form of $\phi'(x)$.

So, $F(y, x) = y^2x + K$

The solution of the exact differential equation should then be F(y, x) = c. K being a constant can be merged with c, so that $y^2x = c$, or $yx = cx^{-1/2}$ where c is arbitrary.

A first order linear differential equation is general written as

$$\frac{dF}{dx} = P(x)y = Q(x) \qquad \text{…. (4)}$$

where P and Q are two functions of x, and that of y. P and Q may also be expressed in other forms as x^2 and e^x. These may also be constants. In the following, we will discuss homogenous differential equations.

Example: Solow's model of economic growth

Consider a production function

$$q = f\left(K, L\right)$$

where q= output, K= capital and L= labour. It is specified that the production function takes the form: $q = AL^{\alpha}K^{1-\alpha}$, where *A* is a positive constant and 0 < *a* < 1. A constant fraction *s* of output is "saved" (with 0 < *s* < 1), and used to augment the capital stock. Thus, the capital stock changes according to the differential equation

$$K^{'}(t) = sAL(t)^{\alpha}K(t)^{1-\alpha}$$

and takes the value K_0 at t = 0. The labour force is L_0 > 0 at t = 0 and grows at a constant rate λ, so that

$$\frac{L'(t)}{L(t)} = \lambda$$

You can solve this model by solving for L, then substitute the value into the equation for $K'(t)$ to get K.

Note that the equation for L is separable, and we can write

$$\frac{dL}{L} = \lambda dt \,.$$

Integrating it we get

$$\log L = \lambda t + C$$
$$or, L = Ce^{\lambda t}$$

Given the initial condition, we have $C = L_0$.

Substituting this result into the equation for $K'(t)$ yields

$$K'(t) = sA(K(t))^{1-\alpha}(L_0 e^{\lambda t})^{\alpha} = \mathrm{sA}\left(\mathrm{L}_0\right)^{\alpha} e^{\alpha\lambda t}\left(K\left(t\right)\right)^{1-\alpha}$$

This equation is separable, and may be written as

$$\mathrm{K}^{\alpha\text{-}1} dK = sA(L_0)^{\alpha} e^{\alpha\lambda t} dt \,.$$

Integrating both sides, we obtain

$$\frac{\mathrm{K}^{\alpha}}{\alpha} = sA(L_0)^{\alpha}\frac{e^{\alpha\lambda t}}{\alpha\lambda} + C,$$

so that

$$\mathrm{K}\left(\mathrm{t}\right) = \left(sA\left(L_0\right)^{\alpha}\frac{e^{\alpha\lambda t}}{\lambda} + C\right)^{\frac{1}{\alpha}}.$$

Given $\mathrm{K}\left(0\right) = K_0$, we conclude that $C = (K_0)^{\alpha} - \frac{sA(L_0)}{\lambda}$.

Thus, $\mathrm{K}\left(\mathrm{t}\right) = \left[sA\left(L_0\right)^{\alpha}\frac{e^{\alpha\lambda t}-1}{\lambda} + (K_0)^{\alpha}\right]^{\frac{1}{\alpha}}$ for all t.

An interesting feature of the model is the emergence of capital-labor ratio. We have

$$\frac{K(t)}{L(t)} = \frac{\left[sA\left(L_0\right)^{\alpha}\frac{e^{\alpha\lambda t}-1}{\lambda} + (K_0)^{\alpha}\right]^{\frac{1}{\alpha}}}{L_0 e^{\lambda t}}$$

for all t.

As $t \to \infty$, $\frac{K(t)}{L(t)}$ converges to $\left(\frac{sA}{\lambda}\right)^{\frac{1}{\alpha}}$.

Homogenous Case

If P and Q are constant functions and if Q is identically equal to zero, equation (4) becomes $\frac{dy}{dx}$ + ay = 0 where a is some constant … (5)

Please note that the constant term '0' can be regarded as in the first degree in terms of y because 0y = 0.

Equation (5) can be written as

$$\frac{1}{y}\frac{dy}{dx} = -a \quad \text{…. (6)}$$

For solution, we write $\frac{dy}{y} = cdx$ (with c = –a) and integrate both the sides, such that $\int \frac{dy}{y} = \int cdx$. The left side of the above gives log y + c_1 for y ≠ 0.

Whereas right side becomes

cx + c_2

Bringing together the result of the left and right sides,

log y + c_1 = cx + c_2

or, log y = cdx + c_3 (combining c_1 and c_2 of both sides)

or, $e^{\log y} = e^{(cx+c_3)}$

or, y = $e^{cx}.e^{c_3}$ = A e^{cx} where A = e^{c_3}

Putting back c = –a,

we get y(x) = Ae^{-ax} where A is arbitrary … (7)

To get rid of the arbitrary constant, set x = 0 in the equation y(x) = Ae^{-ax}, so that

y(0) = Ae^{0} = A.

Thus, y(x) = y(0)e^{-ax} … (8)

In (7), A is an arbitrary constant. The solution, therefore, is a **general solution**. When a particular value is substituted for A, we derived the **particular solution** in (8). There are an infinite number of particular solutions, value of y(0). However, y(0) is important since it can alone satisfy the initial condition. From the feature of giving a definite value to the arbitrary constant, we refer the result in (8) as the definite solution of the differential equation.

Non-homogenous case

When we have a non-zero constant in place of the zero in equation (5) above, it is called a non-homogenous linear differential equation.

Thus, $\frac{dy}{dx} + ay = b$ … (9)

is a non-homogeneous differential equation. The solution of this class of equations has two parts, (a) complementary function (y_c) and (b) particular integral (y_p). Before we proceed to solve equation (9), it will be useful to point out that homogeneous equation (5) is called a reduced equation of (9) and the non-homogenous equation (9) itself is categorised as the complete equation. Moreover, the complementary function (y_c) is the general solution of the reduced equation, whereas the particular integral (y_p) is any of the particular solution of the complete equation.

Solution to non-homogenous differential equation is seen as a sum of the complementary function and the particular integral.

Thus, $y(x) = y_c + y_p$.

We have noted above that y_c is the general solution of the reduced equation. We take the general solution of the homogeneous differential equation (5) above, which was $A e^{-ax}$. Thus, $y_c = A e^{-ax}$.

Let us came to particular integral. Recall that it is any particular solution of the complete equation. Perhaps the simplest possible type of solution we can think of is to take it being some constant (y = k). Taking of as a constant, we get $\frac{dy}{dx} = 0$. Therefore, equation (9) becomes ay = b, or y = $\frac{b}{a}$ where $a \neq 0$.

In that case $y_p = \frac{b}{a}$, we get the general solution to the equation as

$y(x) = A e^{-ax} + \frac{b}{a}$ … (10)

See that the solution remains general. The presence of arbitrary constant A is responsible for it. In order to make it definite, we need to take an initial condition. Setting x = 0, y can be assigned the value y(0) and we get

$y(0) = A + \frac{b}{a}$

or, $A = y(0) - \frac{b}{a}$

Putting this value in (10*)*, the solution becomes

$$y(x) = \left[y(0) - \frac{b}{a} \right] e^{-ax} + \frac{b}{a} \qquad \ldots (11)$$

which is the definite solution as long as $a \neq 0$.

Exercise:

Solve the equation $\frac{dy}{dx} + 2y = 6$ with the initial condition y(0) = 10.

We have a = 2 and b = 6. Hence, according to (11), the solution is

$y(x) = [10 - 3]e^{-2x} + 3 = 7e^{-2x} + 3$

Solution when a = 0

If $\frac{dy}{dx} + ay = b$ has a = 0, then

$$\frac{dy}{dx} = b \qquad \ldots (12)$$

Its general solution is found by integration, i.e., y(x) = bx + c where c = arbitrary constant.

Complementary function: with a = 0

$y_c = Ae^{-ax} = Ae^0 = A$ (A = an arbitrary constant)

Particular Integral

As a = 0, the constant solution y = k does not work and some non-constant solution needs to be tried. Take y = kx so that

$$\frac{dy}{dx} = k.$$

From the complete equation (12) above k = b.

$\therefore\ y_P = bx$

General solution: $y(x) = y_c + y_P = A + bx$ ………. (13)

Example:

Solve the equation $\frac{dy}{dx} = 2,$ with the initial condition y(0) = 5. 0

From (13) above, y(x) = 5 + 2x.

Verification of the Solution

You can check the correct answer of your solution to a differential equation by taking its differentiation. Follow the following two steps:

1) Test that the derivative of the time path is consistent with the given differential equation.

2) Test the definite solution to find that the solution satisfies the initial condition.

Check Your Progress 1

1) Solve the following differential equations.

   a) $(y(t))^2 y'(t) = t + 1$;

   b) $y'(t) = t^3 - t$.

   c) $y'(t) = te^t - t$.

   d) $y(t) = e^{y(t)} y'(t) = t + 1$.

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2) Solve the following differential equation for the given initial value.

   a) $ty'(t) = y(t)(1 - t)$, $(t, y) = (1, 1/e)$.

   b) $(1 + t^3)y'(t) = t^2 y(t)$, $(t, y) = (0, 2)$.

   c) $y(t)y'(t) = t$, $(t, y) = (\sqrt{2}, 1)$.

   d) $e^{2t} y'(t) - (y(t))^2 - 2y(t) - 1 = 0, (t, y) = (0, 0)$.

3) Find y_c, y_p , the general solution and definite solution of the equation and check its validity:

   $$\frac{dy}{dx} + 4y = 12; y(0) = 2$$

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7.4.2 Solving Linear First-order Differential Equations

A **linear first-order differential equation, in general,** takes the form

$y'(t) + a(t)y(t) = b(t)$ for all t and with a and b representing functional forms.

Coefficient of *y*(*t*) constant

Consider the case in which $a(t) = a \neq 0$ for all t, so that

$y'(t) = a(t)y(t) = b(t)$ for all t.

If the left-hand side were the derivative of some function and we could find the integral of b then we could solve the equation by integrating each side. If we multiply both sides by $g(t)$ for each t, then

$g(t)y'(t) + ag(t)y(t) = g(t)b(t)$ for all t.

See that the left-hand side of *this* equation to be the derivative of a product of the form $f(t)y(t)$ provided we have $f(t) = g(t)$ and $f'(t) = ag(t)$. See that if $f(t) = e^{at}$, then $f'(t) = ae^{at} = a f(t)$.

Thus if we set $g(t) = e^{at}$, so that we have

$e^{at}y'(t) + ae^{at}y(t) = e^{at}b(t)$ and the integral of the left-hand side is $e^{at}y(t)$. We get the solution of the equation as

$e^{at}y(t) = C + \int e^{as}b(s)ds$

or, $y(t) = e^{-at}\left(C + \int e^{as}b(s)ds\right)$

So, the general solution of the differential equation

$y'(t) + ay(t) = b(t)$ for all t,

where a is a constant and b is a continuous function, is given by

$y(t) = e^{-at}\left(C + \int e^{as}b(s)ds\right)$ for all t.

Because multiplying the original equation by e^{at} allows us to integrate the left-hand side, we call e^{at} an **integrating factor**.

If $b(t) = b$ for all t then the solution simplifies to

$y(t) = Ce^{-at} + b/a$

Looking at the original equation we see that $y'(t) = 0$ if and only if $y(t) = b/a$. Thus $y = b/a$ is an equilibrium state.

For the initial condition $y(t_0) = y_0$ we have $y_0 = Ce^{-at_0} + b/a$ so that $C = \left(y_0 - b/a\right)e^{at_0}$. The solution of the difference equation is given by

$$y(t)=\left(y_0-\frac{b}{a}\right)e^{a(t_0-t)}+\frac{b}{a}.$$

As $t\to\infty$, $y(t)$ converges to b/a if $a > 0$, and grows without bound if $a < 0$ and $y_0 \neq b/a$. That is, the equilibrium is stable if $a > 0$ and unstable if $a < 0$.

Example:

The demand function is D(p) = a – bp and that of the supply is S(p) = α + β*p*, where *a*, *b*, α, and β are positive constants. If the speed at which the price changes is proportional to the difference between supply and demand, find the equilibrium price and examine its stability.

Since $p'(t)=\lambda(D(p)-S(p))$ with $\lambda>0$ from the supply and demand functions we have

$p'(t)+\lambda(b+\beta)p(t)=\lambda(a-\alpha)$. Consequently, the general solution of this differential equation is

$p(t)=Ce^{-\lambda(b+\beta)t}+(a-\alpha)/(b+\beta)$ and the equilibrium price is $(a-\alpha)/(b+\beta)$. Since $\lambda(b+\beta)>0$, the equilibrium derived is stable.

Check Your Progress 2

1) Find the general solution of $y'(t)+(1/2)y(t)=1/4$. Determine the equilibrium state and examine its stability.

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2) Find the general solution of the differential equation $y(t)-3y(t)=5y'(t)-3y(t)=5$ if the initial value is given as $y(0)=1$.

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3) Solve the differential equation, $ty'(t)+2y(t)+t=0$ for $t\neq 0$.

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7.4.3 Solving Second-order Differential Equations

General form

A **second-order ordinary differential equation** consists of time as the independent variable with the dependant variable y with its first and second derivatives. Consider for example an equation $G(t, y(t), y'(t), y''(t)) = 0$ for all t such that we can write it in the form

$$y''(t) = F(t, y(t), y'(t)) .$$

Equations of the form y″(t) = F (t, y′(t))

Take an equation of form

$$y''(t) = F(t, y'(t)),$$

in which $y(t)$ does not appear. See that can be reduced to a first-order equation if we take $z(t) = y'(t)$.

Example:

Consider *Arrow-Pratt measure of relative risk aversion,*

$\rho(w) = \frac{-wu''(w)}{u'(w)}$ where $u(w)$ is postulated a function for wealth w.In such a formulation, if we consider two utility functions, u and v, then greater risk-aversion of the former is assumed whenever $\rho_u(w) > \rho_v(w)$.

Find the utility function that has a degree of risk-aversion independent of the level of wealth? Or, for what utility functions u do we have an equation

$a = \frac{-wu''(w)}{u'(w)}$ for all w?

Note that we have a second-order differential equation in which the term $u(w)$ does not appear. If we define $z(w) = u'(w)$, then

$$a = \frac{-wz'(w)}{z'(w)}$$

or, $az(w) = -wz'(w)$.

The equation becomes separable and we can write as

$$a\frac{dw}{w} = -\frac{dz}{z}.$$

Consequently, its solution is given by

$$a\ln w = -\ln z(w) + C$$

or, $a \ln w = \ln z(w) + C$,

or, $z(w) = Cw^{-a}$

Since we have taken $z(w) = u^{'}(w)$, to get *u* by integrating

u(w)=Clnw+B if a =1

and

$\frac{Cw^{-a}}{a-1} + B$ if $a \neq 1$.

Thus, when a takes this form we have a utility function with a constant degree of risk-aversion.

Linear second-order equations with constant coefficients

A **linear second-order differential equation with constant coefficients** takes the form

$$y^{''}(t) + ay^{'}(t) + by(t) = f(t)$$

for constants *a* and *b* and a function f. The above equation is **homogeneous** when if $f(t) = 0$ for all *t*.

Let us call $y^{''}(t) + ay^{'}(t) + by(t) = f(t)$ as the "original equation" and assume that y_1 as its solution. For any other solution of this equation *y*, define $z = y - y_1$.

Since $y - y_1$ can be writen as

$$\left[y^{''}(t) + ay^{'}(t) + by(t)\right] - \left[y^{''}_1(t) + ay^{'}_1(t) + by_1(t)\right] = f(t) - f(t) = 0,$$

z is a solution of the homogeneous equation

$$y^{''}(t) + ay^{'}(t) + by(t) = 0$$

Further, for every solution *z* of the homogeneous equation, $y_1 + z$ is also a solution of original equation. Therefore, as has been discussed above in first order non-homogenous equation case, solutions of the original equation may be found by

- a particular solution of the equation and
- adding to it the general solution of the homogeneous equation.

Finding the general solution of a homogeneous equation

Recall that the solution derived in case of first-order homogenous equation was of the form $y(t) = Ae^{rt}$. Therefore, we can write $y^{'}(t) = Ae^{rt}$ and $y^{''}(t) = Ae^{rt}$. Substituting these into $y^{''}(t) + ay^{'}(t) + by(t)$

We get

$r^2 Ae^{rt} + arAe^{rt} + bAe^{rt}$

$= \mathrm{Ae}^{\mathrm{rt}}(r^2 + ar + b).$

Thus, for $y(t)$ to be a solution of the equation we need

$r^2 + ar + b = 0.$

This equation is knows as the **characteristic equation** of the differential equation.

Let us look at the solutions offered by the characteristic equaion. If

- $a^2 > 4b$, then there are two distinct real roots, say r_1 and r_2. We have both $y(t) = A_1 e^{r_1 t}$ and $y(t) = A_2 e^{r_2 t}$ as solutions the equation for any values of A_1 and A_2. Hence, also $y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$ is a solution. It can be shown that *every* solution of the equation takes this form;
- $a^2 = 4b$, then the characteristic equation has a single real root. The general solution of the equation is $(A_1 + A_2)e^{rt}$, where $r = -\frac{1}{2}a$;
- $a^2 < 4b$, then the characteristic equation has complex roots. Derivation of the results on these roots will be taken up in greater details in Unit 8 and we will present here the general solution of the equation as

  $(\mathrm{A}_1\cos(\beta \mathrm{t}) + A_2 \sin(\beta t))e^{\alpha t},$

where $\alpha = -\frac{a}{2}$ and $\beta = \sqrt{\left(b - \frac{a^2}{4}\right)}$. We will express this solution alternatively as $\mathrm{Ce}^{\alpha t}\cos(\beta t + \omega)$, where the relationships between the constants C, ω, A_1, and A_2 are $\mathrm{A}_1 = \mathrm{C}\cos\omega$ and $\mathrm{A}_2 = -\mathrm{C}\sin\omega$.

Solution of a second order nonhomogeneous equation

We follow a procedure similar to the one in case of first order equation. In the second order equation take a linear combination of $f(t)$ and its first and second derivatives to try for solution that satisfies the equation. If, for example,

- $f(t) = 3t - 6t^2$, then examine the values of *A*, *B*, and *C* such that $A + Bt + Ct^2$ is a solution;
- $f(t) = 2\sin t + \cos t$, find values of *A* and *B* such that $f(t) = A\sin t + B\cos t$ is a solution;
- $f(t) = 2e^{Bt}$ for some value of *B*, find a value of *A* such that Ae^{Bt} is a solution.

Stability of solutions of second order homogeneous equation

Consider the above homogeneous equation

$$y''(t) + ay'(t) + by(t) = 0.$$

If $b \neq 0$, this equation has a single equilibrium, viz., 0. That is, the only constant function that is a solution is equal to 0 for all *t*. We will consider three possible forms of the general solution of the equation to evaluate the stability of such an quilibrium.

Case I: Characteristic equation has two real roots

If r_1 and r_2, are the two roots of the characteristic equation, then the general solution of the equation is $y(t) = Ae^{r_1 t} + Be^{r_2 t}$. The equilibrium is stable if and only if $r_1 < 0$ and $r_2 < 0$.

Case II: Characteristic equation has a single real root

With a single root (say *r*), the characteristic equation is in stable equilibrium if and only if this root is negative. Note that if $r < 0$ then for any value of *k*, $t^k e^{rt}$ converges to 0 as $t \to \infty$.

Case III: Characteristic equation has complex roots

When the characteristic equation has complex roots, the form of the solution of the equation is $\mathrm{Ae^{at}} \cos(\beta t + \omega)$, where $\alpha = -\frac{a}{2}$, the *real* part of each root. The equilibrium will be stable if and only if the real part of each root is negative.

On the basis of the above results we can sya that the stability of the equilibrium is ensured if and only if the real parts of both roots of the characteristic equation are negative. A bit of algebra shows that this condition is equivalent to $a > 0$ and $b > 0$.On the other hand, if $b = 0$, then every number is an equilibrium, and none of these equilibria is stable.

Check Your Progress 3

1) Solve the differential equation $y''(t) + y(t)' - 2y = -10$ with initial conditions $y(0) = 12$ and $y'(0) = -2$.

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2) Solve the differential equation $y''(t) + 6(t) + 9y = 27$ with initial conditions $y(0) = 5$ and $y'(0) = -5$.

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7.4.4 Economic Applications: Examples

a) **The Harrod-Domar Analysis of Steady Growth**

Consider a macroeconomic model consisting of the following equations:

$$S(t) = sY(t),\ 0 < s < 1$$

$$I(t) = v\frac{dy}{dx},\ v > 0$$

$$S(t) = I(t)$$

where Y, S, I stand for the rates of flow of national income, planned saving and planned investment at any point of time t. The first equation says that a constant fraction s of income is saved in each time period. The second equation represents the acceleration theory of investment in which induced investment is proportional to the rate of change of income (v is a constant of proportionality). There is no autonomous investment. For dynamic equilibrium, we need equally between saving and investment as each period. This is the significance of the final equation. The three equations lead to a simple differential equation

$$\frac{dY}{dt} - \frac{s}{v}Y = 0 \qquad \text{… (14)}$$

By formula (10), the general solution is

$$Y(t) = Ae^{(s/v)t}$$

Suppose it is known that $Y = Y_0$ at $t = 0$. Then $A = Y_0$ so that the equation becomes

$$Y(t) = Y_0\, e^{(s/v)t} \qquad \text{… (15)}$$

This solution gives the behaviour of national income over time in dynamic equilibrium. For any variable x that changes with time, its **rate of growth** at any point in time is defined to be $\left(\frac{1}{x}\frac{dx}{dt}\right)$, the rate of change of x divided by (as a proportion of) the value of x at that point of time. It is clear from equation (14) that in this model the rate of growth of national income $\left(\frac{1}{Y}\frac{dY}{dt}\right)$

assumes the constant value $\frac{s}{v}$. In other words, national income grows at the constant rate $\frac{s}{v}$ in dynamic equilibrium.

It follows as a corollary from our discussion that if a variable x is changing at a constant rate g over time (growing if g > 0, decaying if g < 0), then its time path is given by $x = x_0 e^{gt}$ where x_0 is the value of x at initial or base period (t = 0).

Our next application is taken from microeconomics. It will help us to understand the notion of stability.

b) **The Dynamics of Price in a Single Market**

Suppose the demand and supply functions for a particular commodity are given by

$$D_t = a - bP_t$$

$$S_t = c + bP_t;\ a, b, c, d > 0,\ a > c$$

The first equation tells us that demand D_t in a particular period t is a decreasing linear function of price prevailing in that period P_t. The second equation, the supply function, has a similar interpretation.

The equilibrium price in this context has the property that it (i) clears the market in each period and (ii) does not change over time. Let us denote it by P^*. (Note that since it is constant through time there is no t-subscript). Writing P^* for P_t in the demand and supply equations and setting $D_t = S_t$ we obtain the expression for the equilibrium price

$$P^* = \frac{a-c}{b+d}$$

The restriction a> c ensures that P^* is positive. Now comes an important point. Knowledge of the equilibrium price tells us nothing about the behaviour of price out of equilibrium. In other words, although we know P^* we do not know what happens when in any period t the price Pt is not equal to P^*. Does price rise or fall or fluctuate in some unpredictable manner? To answer questions of this type precisely we have to introduce a dynamic **adjustment rule** for price. It seems natural and sensible to assume that price will tend to rise if demand exceeds supply and fall if demand falls short of supply and stay unchanged if demand and supply just balance in any period. This type of adjustment is incorporated in the analysis through a simple liner relationship.

$$\frac{dP}{dt} = \theta(D_t - S_t),\ \theta > 0 \qquad \dots (16)$$

Since θ is a positive constant, this tells us that

i) $\frac{dP}{dt} = 0$ or P rises if $Dt > S_t$

ii) $\frac{dP}{dt}$ = 0 or P falls if $Dt < S_t$ and

iii) $\frac{dP}{dt}$ = 0 or P stays unchanged if $Dt = S_t$

Substituting the demand and supply functions we obtain the differential equation for price

$$\frac{dP}{dt} = \theta(b + d)\, P = \theta(a - c).$$

This, being of the form (9), has the solution

$$P = Ae^{-\theta(b + d)t} + \frac{a-c}{b+d}.$$

Suppose it is known that $P = P_0$ at t = 0. Then from the above solution

$A = \left(P_0 - \frac{a-c}{b+d}\right)$. But $\frac{a-c}{b+d} = P^*$, the equilibrium price. So the solution can be written as

$$P = (P_0 - P^*)\, Ae^{-\theta(b + d)t} + P^* \qquad \dots (17)$$

This completely describes the time path of price in the market characterised by the given demand and supply curves and the adjustment rule (16). Note that the time path (for any given initial price P_0) is determined by the demand and supply parameters (a, b, c, d) and the coefficient of adjustment θ. The dependence of the solution on θ brings out clearly the importance of adjustment rules in dynamics.

Now let us take up the question of **stability**. Suppose that the initial price P_0 is not the equilibrium price P^*, that is $(P_0 - P^*) \neq 0$. The system is stable if price tends to approach the equilibrium price P^* as time passes, that is if $\underset{t\to\infty}{Lt}\, P(t)=P^*$.

It is clear that since $\theta > 0$ and (b + d) > 0 the term $e^{-\theta(b + d)t}$ (and hence the first term of (16) will tend to zero as t tends to infinity, so that P will indeed converge to P^* and we have a system that is dynamically stable.

7.5 LET US SUM UP

Economic models with a temporal dimension involve relationships between the values of variables at a given point in time and the changes in these values over time. Solution to such problems are attempted by taking time is as a continuous (or discrete) variable. In this unit, we have discussed some of the basic tools of dynamic analysis – the indefinite integral, the definite integral and differential equations. In the process, we have learnt the application of such tools in solving problems related to economic dynamics.

7.6 KEY WORDS

Consumer's Surplus: This notion was introduced by Alfred Marshall to measure the net benefit that a consumer enjoys from his act of purchasing a particular commodity in the market. It is defined in terms of the excess of the consumer's total willingness to pay in units of money over his actual expenditure.

Definite Integral: The definite integral of the function *f*(x) over the interval (a, b) is expressed symbolically as $\int_a^b f(x)dx$, read as "integral of *f* with respect to x from a to b. The smaller number a is termed the **lower limit** and b, the **upper limit** of integration. Geometrically, this definite integral denotes the area under the curve representing *f* (x) between the points x = 0 and x = b. Note that the definite integral $\int_a^b f(x)$ is a number.

Differential Equations: Differential equations are equations involving the derivatives (or differentials) of unknown functions. Solving a differential equation means finding a function that satisfies that equation.

Economic Dynamics: Dynamics is essentially concerned with change and the effects of change on the behaviour of variables over time. Economic dynamics deals with economic variables like national income, price, etc. The task of dynamics is to consider the actual process of transition from the initial pre-change position to the final equilibrium.

Equilibrium: If an initial value has a solution that is a constant function (i.e., independent of *t*), then the value of the constant is called an equilibrium state or **stationary state** of the equation.

Improper Integral: It is a special type of definite integral. When one of the limits of integration is + ∞ or – ∞, a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits.

Indefinite Integral: The indefinite integral is basically reverse differentiation. To differentiate means to find the rate of change (derivative) of a given function. Indefinite integration reverses the process and finds the unknown function whose rate of change (derivative) is given.

Initial Value: To solve the differential (or difference) equation by specifying the value of y or the value of its derivatives at any value of *t*.It may not necessarily be the "first" value.

Stable Solution : If, for all initial conditions, the solution of the differential (or difference) equation converges to the equilibrium as *t* $t \to \infty$, then the equilibrium is **stable**.

7.7 SOME USEFUL BOOKS

Archibald, G.C. and R.G. Lipsey, 1983, *An Introduction to a Mathematical Treatment of Economics* (Third Edition), ELBS London, Chapters 12, 13 and 14.

Baumol, W.J., 1974, *Economic Dynamics* (Second Edition), Macmillan, New York, Chapter 14.

Chiang, Alpha C., 1983, *Fundamental Methods of Mathematical Economics* (Third Edition) McGraw Hill, International Students Edition Chapters 13, 14.

IGNOU, 1990, *MTE-01: First Elective Course in Mathematics* (Block 3: Integral Calculus).

7.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) a) $y(t)=(\frac{3}{2})t^2+3t+3C)^{\frac{1}{3}}$

b) $y(t)=\frac{t^4}{4}-\frac{t^2}{2}+C$

c) $y(t)=te^t-e^t+\frac{t^2}{2}+C$

d) $y(t)=\log(\frac{1}{2})t^2+t+C$

e) The equation is separable:

b $\int ye^y dy=\int\frac{1}{t}dt$. So integrating by parts on the left to get ye^y-e^y = $\ln t + C$. Thus the solution is defined by the condition $\left(y(t)-1\right)e^{y(t)} = \ln t + C$.

f) The equation is separable:

$\int(1/(4y+1))dy = \int t\,dt$,

so that $(1/4)\ln(4y+1) = (1/2)t^2 + C$,

or, $y(t) = C\exp(2t^2) - 1/4$.

2) a) $y(t) = Cte^{-1}$; C = 1.

b) $y(t) = C(1+t^3)1/3$; C = 2.

c) $y(t) = \sqrt{(t^2+C)}$; C = -1.

d) $y(t) = \frac{(2 - C - e^{-2t})}{(C + e^{-2t})}$; C = 1.

3) $y(x) = e^{-4x}+3$

Check Your Progress 2

1) $y(t) = Ce^{-\frac{t}{2}} + \frac{1}{2}$. Equilibrium: $y^* = 1/2$; stable.

2) $y(t) = Ce^{3t} - \frac{5}{3}$; C=8/3

3) $y(t) = \left(\frac{1}{t^2}\right)\left[C - \frac{t^3}{3}\right] = \frac{C}{t^2} - \frac{t}{3}$

Check Your Progress 3

1) $y(t) = 4e^{t} + 3e^{-2t} + 5$

2) $y(t) = 2e^{-3t} + te^{-3t} + 3$

7.9 EXERCISES

1) If the rate of change of y with respect to x is 2x and y = 4 when x = 1, find y as a function of x.

2) The rate of change of y with respect to x is $(0.8x - 0.6x^2)$ and y = 0 and x = 0. Find y as a function of x.

3) Let the consumer's demand function be P = 20 – 2q.

Calculate the consumer's surplus for P = 8. Is it larger or smaller than the CS for P = 47?

4) At the rate of interest of 4 per cent a year, what is the present value of Rs.1000 available 2 years later?

5) A piece of land yields a constant rent of Rs.1000 per year. Find its market value if the rate of interest is 10 per cent per year.

6) Solve the equation $\frac{dy}{dx} - 5y = -25$ with y(0) = 6

7) Explain the dynamics of price adjustment process in a single market. What happens if $\theta < 0$?

8) Solve the equation $\frac{dy}{dx} + 4y = 0$, with initial condition y(0) = 1.

9) Find y_c, y_p, the general solution and definite solution of the equation and check its validity $\frac{dy}{dx} + 4y = 12$; y(0) = 1.

10) Solve $\frac{dy}{dx} - 5y = 0$; y(0) = 6 and check its validity.

11) Solve the differential equation $y''(t)+3y'-4y(t)=12$ with initial condition $y(0)=4$, $y'(0)=2$. Check the stability of the solution.

12) Find the particular solution of the defferential equation

$$y''(t)+y'(t)-2y=-10.$$

13) Solve the differential equation $y''(t)-4y'(t)+4y(t)=5$ with the initial conditions $y(0)=4$ and $y'(0)=6$.

Answer or Hints to Exercises

1) $\dfrac{dy}{dx}=2x$

or $\int dy=\int 2xdx=2\int xdx=\dfrac{2x^2}{2}+c=x^2+c$

or y = x^2 + c

Now y = 4 when x = 1 so that 4 = 1 + c or c + 3.

$\therefore$ y = f(x) = x2 + 3

2) $\dfrac{dy}{dx}=0.8x-0.6x^2$

or $\int dy=\int (0.8x-0.6x^2)dx$

= $\int 0.8xdx-\int 0.6x^2dx$

$=0.8\int xdx-0.6\int x^2dx$

$\therefore$ y = $0.8\dfrac{x^2}{2}-0.6\dfrac{x^3}{3}+c$

or y = $0.4x^2 - 0.2x^3$ + c

Now y = 0 for x = 0 $\Rightarrow$ c = 0

$\therefore$ y = $0.4x^2 - 0.2x^3$

3) P = 20 – 2q = f(q)

For P =8, 12 = 2q or q = 6.

$\therefore$ Pq = 48 = total expenditure when P = 8.

$$\int_0^6 f(q)dq=\int_0^6 (20-2q)dq$$

$$= 20\int_0^6 dq - \int_0^6 2q\,dq$$

$$= 20\int_0^6 dq - 2\int_0^6 q\,dq$$

$$= 20[q]_0^6 - 2\left[\frac{q^2}{2}\right]_0^6 = 120 - 36 = 84.$$

$\therefore$ Consumer's surplus $= \int_0^8 f(q)dq - Pq = 84 - 48 = 36.$

Now, for P = 4, q = 8.

CS $= \int_0^8 (20 - 2q)dq - 32 = 64$

$\therefore$ Consumer's surplus increases as the price of the commodity falls.

4) Let A = amount; P = principal, r = rate of interest, n = number of years.

Then applying the compound interest formula,

$A = P(1 + r)^n$,

Here A = 100, r = 0.04, n = 2, P = ?

Hence P = $\dfrac{100}{(1+0.04)^2} = \dfrac{100}{(1.04)^2}$

5) Let Y = market value of land

$$Y = \int_0^\infty Re^{-rt}dt$$

$= \dfrac{R}{r}$ (For the steps which you have to do, refer to Example of Economic Applications of the Definite Integral). Here R = 1000, r = 0.1

$Y = \dfrac{1000}{0.1} = 10{,}000$

6) $\dfrac{dy}{dx} - 5y = -25$

Here m = –5, k = –25

$\therefore$ y = $Ae^{-mx} + \dfrac{k}{m}$ becomes

$y = Ae^{5x} + \frac{-25}{-5} = Ae^{5x} + 5.$

Now, $y(0) = 6 \Rightarrow Ae^0 + 5 = 6$ or A.1 = 1 or A = 1.

$\therefore$ $y = e^{5x} + 5$, is the answer.

7) See Section 7.4.1 Example.

   If $\theta < 0$, the system cannot attain a stable equilibrium, i.e., the system is unstable. You reason why.

8) $y(x) = [1 - 0]e^{-4x} + c = e^{-4x}$

9) $y(x) = -e^{-4x} + 3$

10) $y(x) = 6e^{5x}$

11) The roots of the characteristic equation are 1 and –4. A particular integral is $y(t) = -3$. Thus, the general solution is

$$Ae^{t} + Be^{-4t} - 3.$$

   For the given initial conditions we have $A = 6$ and $B = 1$. The general solution is unstable.

12) $y_p = \frac{b}{a}t$

13) $y(t) = A_1 e^{2t} + A_2 te^{2t} + \frac{5}{4} with A_1 = \frac{11}{4} and A_2 = \frac{1}{2}$

UNIT 8 DIFFERENCE EQUATIONS AND APPLICATIONS IN ECONOMIC DYNAMICS

Structure

8.0 OBJECTIVES

After going through this unit you should be able to:

- solve problems of economic dynamics where the time variable takes only discrete values.

8.1 INTRODUCTION

A **difference equation** is used to solve the values of an unknown function y(x) for different discrete values of x. We obtain a function y(x) such that it satisfies the equation for all values of x. In order to understand the process of formulation of the difference equation, you may recall the discussion on differential equation presented in the preceding unit. See that difference and differential equations are exactly analogous with the only difference that the former applies when the independent variable takes only discrete values, whereas the latter when it is continuous.

8.2 DIFFERENCE EQUATIONS IN ECONOMICS

To get an idea about how difference equations may come up in economics consider the case where it is known that the national income y of a particular country has been growing at a constant rate g over (say) a ten year period starting from some base year. The rate of growth of y at any period t may be represented as $\left(\frac{y_t - y_{t-1}}{y_{t-1}}\right)$. Note that this is the expression that gives the rate of growth of y at a particular point in time. In contrast, you have seen the expression $\frac{dy(t)/dt}{y}$ when y was treated as a continuous variable. Treating y as discrete, we find the numerator as the increment of income in current period (period t) over the level attained in the immediately preceding period (period t – 1). The ratio of this to the income y_{t-1} of the preceding period gives the current rate of growth. Since we have said that rates of growth of income, g, is constant (i.e., independent of t) over ten year's interval, it can be written as:

$$\frac{y_t - y_{t-1}}{y_{t-1}} = g, \; t, = 1, 2, 3, 4, \ldots, 10$$

or, $y_t = (1 + g)\, y_{t-1}$, t = 1, 2, …, 10…………………………………….. (1)

Equation (1) relates the values of the variable y at two distinct periods t and (t – 1). It is an example of a difference equation.

There is a one-period lag in the values of the relevant variable (y_t and y_{t-1}). Therefore, it is an example of a **first order** difference equation. The **order** of a difference equation is determined by the **maximum** number of periods lagged. Some examples of difference equations are given below with the orders noted.

$y_{t-3} - 3y_{t-4} = 0$	order 1.
$y_t = a(y_{t-1} - y_{t-2}) + 10$	order 2.
$\log y_{t+9} - y_{t+7}\,(y_{t+6})^3 + 6y_t = 0$	order 9.
$18\, y_{t+4} - y_t = 2^t - 5^{5+1}$	order 4.
$y_{t+3} + ay_{t+1} = by_{t-1} + c$	order 4

Consider a difference equation of the following form:

$y = a_1 y_{t-1} + a_2 y_{t-2} + \ldots. + a_n y_{t-n} + b$

where $a_1, a_2, \ldots, a_n$ and b are constants.

This is called an **n^th^ order linear constant coefficient** difference equation (assuming $a_n \neq 0$, otherwise the order will be less than n). It is linear because the dependent variable y is not raised to any power and there are no product terms, constant coefficients because $a_1, \ldots, a_n$ are constants and do not change with t. This equation will be **homogeneous** if b = 0. If $b \neq 0$, then it is **non-homogenous**. In this unit, we shall work only with difference equations of this special type of orders one and two (n = 1, 2).

8.3 SOLVING FIRST ORDER DIFFERENCE EQUATIONS

In solving a difference equation, we find a time path y(t) from a given initial condition. As pointed out above, a first order difference equation takes the form

$y_t = f(t, y_{t-1})$ for all *t*.

We can solve such an equation by successive calculation,also called recursive method , taking the initial value of y (say y_0) as given. Thus,

$y_1 = F(1, y_0)$
$y_2 = F(2, y_1) = f(2, f(1, y_0))$
and so on.

Note that given any value y_0, there exists a unique solution path y_1, y_2, ...

However, resorting to calculation of the solution through such a method doesn't tell us much about the properties of the solution. We should have a general formula, which exists if the form of f is simple.

Let us start with a first-order linear difference equation with constant coefficient. It takes the form

$y_t = ay_{t-1} + b_t$ where b_t for $t = 1$, ... are constants.

When the recursive method is used, you will see a pattern as follows:

$$y_t = ay_0 + \sum_{k=1}^{t} a^{t-k} b_k \quad \text{..........................(2)}$$

and such an equation has a unique solution path.To check that we get the unique solution from the above formulation, verify that it satisfies the original equation.

Since we have

$$ay_{t-1} + b_t = a\left(a^{t-1}y_0 + \sum_{k=1}^{t-1} a^{t-k} b_k\right) + b_t$$

$$= a^t y_0 + \sum_{k=1}^{t-1} a^{t-k} b_k + b_t$$

$$= a^t y_0 + \sum_{k=1}^{t} a^{t-k} b_k$$

$$= y_t\,,$$

so that the solution obtained is correct.

Taking the equation (2), we can examine the special case of

$b_k = b$ for all $k = 1$……..

We have

$$y_t = a^t y_0 + b\sum_{j=0}^{t-1} a^{t-1}$$

Making use of the result of geomtric series summation, the term $\sum_{j=0}^{t-1} a^{t-1}$ may be expanded as $1 + a + a^2 + ... + a^{t-1}$ to give

$1 + a + a^2 + ... + a^{n-1} = (1-a^n)/(1-a)$.

if $a \neq 1$. Thus we have

$y_t = a^t y_0 + b\cdot(1 - a^t)/(1 - a)$

if $a \neq 1$.

For any given value y_0, the unique solution of the difference equation

$y_t = ay_{t-1} + b$,

where $a \neq 1$, is

$y_t = a^t(y_0 - b/(1 - a)) + b/(1 - a)$.

Equilibrium or Stationary Value

For a given value y_0, the value of y_t changes with *t*. But there may be some value of y_0 for which y_t doesn't change. Such a solution exists if

$y^* = b/(1 - a)$

and y_t is constant, equal to $b/(1 - a)$.

We call y* the **equilibrium** value of y and rewrite the solution as

$y_t = a^t(y_0 - y^*) + y^*$.

Example: Solve $y_{t+1} = \alpha y_t + \beta$, ……………………..…..(3)

where α and β are constants.

Look for a stationary or equilibrium value of y_t over time which can be repeated for any t consistently satisfying the above equation. May be you consider $\bar{y}$ as an equilibrium value of y_t such that

$$\bar{y} = \alpha\bar{y} + \beta$$
$$or, \bar{y} = \frac{\beta}{1-\beta}$$

To understand the above example, we need to remember the dynamic multiplier.

Write

$C_t = \alpha y_{t-1} + \beta$ ……………………………………..(4)

Let the investment be fixed at $\bar{I}$ for every t so that we have

$$Y_t = C_t + I_t = C_t + \bar{I}$$
$$= \alpha Y_{t-1} + (\beta + \bar{I})$$
$$= \alpha Y_{t-1} + \beta^{'}$$

where $\beta^{'} = \beta + \bar{I}$

Use the above relation (4) we have

$$Y_{t+1} = \alpha Y_t + \beta^{'}$$

If an equilibrium income $\bar{Y}$ is found, the solution can be written as

$$\bar{Y} = \alpha\bar{Y} + \beta^{'}$$
$$= \frac{\beta^{'}}{1-\alpha} = \frac{\beta + \bar{I}}{1-\alpha}$$

Note that $\frac{1}{1-\alpha}$ is the Keynesian multiplier.

It is important to remember that we have solved equation (3) for the stationary level of y_t i.e., $\bar{y}$. There is no guarantee that the actual path of y converges to $\bar{y}$. In case y_t approaches $\bar{y}$, then

$$(y_t - \bar{y}) \to 0.$$

If these values of y_t and y_{t+1} hold, we can write

$g_t = y_t - \bar{y}$ ………………..(5)

Since y_t and $\overline{y}$ satisfy (3), we have

$$y_{t+1} = \alpha y_t + \beta \text{ and}$$
$$\overline{y} = \alpha \overline{y} + \beta$$

Thus,

$$y_{t+1} - \overline{y} = \alpha\left(y_t - \overline{y}\right).$$

From (5),

$$g_t = y_t - \overline{y}$$
$$\text{or, } g_{t+1} = y_{t+1} - \overline{y}$$
$$or, \ g_{t+1} = y_{t+1} - \overline{y}$$
$$or, g_{t+1} = \alpha g_t \text{..................(6)}$$

Since

$$g_{t+1} = \alpha g_t$$
$$g_t = \alpha g_{t-1}$$

.

.

.

.

$$g_1 = \alpha g_0$$

Substituting backward,

$$g_{\&+1} = \alpha^2 g_{t-1} = \alpha^3 g_{t-2} \quad \text{..................}$$

we get

$$g_{t+1} = \alpha^{t-1} g_0$$
$$or\ g_t = \alpha^t g_0 \text{ for } t = 0, 1, 2 \text{ ....}$$

Thus, any difference equation of the form $y_t = \alpha y_{t-1}$ has a solution $y_t = \alpha^t y_0$, where y_0 is the value of y at some chosen initial point.

General Solution

Suppose we intend to solve the equation

$$y_{t+1} + ay_t = c \text{................ (7)}$$

Its general solution will be consisting of particular solution $\left(y_p\right)$ and complementary function $\left(y_c\right)$, i.e., $y_g = y_p + y_c$. In this approach, the y_p component represents the inter-temporal equilibrium level of y while that of y_c gives the deviations if the time path from that equilibrium. The solution is called general solution due to the presence of an arbitrary constant. In order to get a definite solution, we need an initial condition.

Let us work with complementary function. From (7), we get its reduced from as

$$y_{t+1} + ay_t = 0 \text{ ....(8)}$$

It is seen above that $y_t = \alpha^t y_o$ is a solution to the difference equation. In that case we have $y_{t+1} = a^{t+1} y_0$ as well. We modify this and rewrite

$y_t = Ab^t$ and $\mathrm{y}_{\mathrm{t+1}} = Ab^{t+1.}$

Substitution of these into (8) gives

$$Ab^{t+1} + aAb^t = 0$$

$$or, Ab^t \left(b + a\right) = 0$$

$$or, (b + a) = 0$$

$$or, b = -a$$

We must have $b = -a$ in the trial solution such that the complementary solution can be written as

$$y_c = Ab^t = A\left(-a\right)^t.$$

Particular solution needs to be recasted such that it is in agreement with the general solution. Consider the simplest value of y. If y_t has an equilibrium value k such that it remains constant overtime, we have $y_t = k$ as well as $y_{t+1} = k.$ Substitution of these values to the trial solution gives

$$k + a_k = c$$

$$or,\ \mathrm{k} = \frac{\mathrm{c}}{1+\mathrm{a}}$$

Since the value, k, satisfies the equation, the particular solution can be written as

$$y_p = k = \frac{c}{1 + a} \quad \text{for} \quad a \neq 0$$

In case $a = -1$, however, the particular solution is not defined and some other solution of (7) must be searched for.

Substituting k into (7), we get

$$k\left(t + 1\right) + ak_t = c.$$

$$or,\ \mathrm{k} = \frac{\mathrm{c}}{\mathrm{t+1+a_t}} = C \text{ and } \mathrm{y_p} = C_{t.}.$$

The general solution can now be written in one of the following forms:

$$y_{tt} = A(-a)^t + \frac{C}{1+a} \quad \text{if} \quad a \neq -1$$

$$or,\ y_t = A(-a)^t + C_t = A + C_t \ \text{if } a = -1.$$

Notice that the solution above still remains indeterminate. This is due to the presence of arbitrary constant A. We have to take the help of initial condition $(y_t = y_o)$ for eliminating it. Thus, taking t =0, we have

$$y_0 = A + \frac{C}{1+a}$$

$$or, A = y_9 - \frac{C}{1+a}$$

The definite solution therefore, becomes

$$y_t = \left(y_0 - \frac{C}{1+a} \right)(-a)^t + \frac{C}{1+a} \quad \text{for } a \neq -1$$

$$or, y_t = y_o + C_t \quad \text{for } a = -1$$

8.3.1 Behaviour of Solutions of First Order Equations

The solution of a difference equation gives an expression for the relevant variable as an explicit function of time. In other words, a time path of the variable is obtained. To investigate the nature of this time path of a solution of the first order equation, we write the solution for a ≠ 1.

The behavior of the solution path depends on the value of *a*.

$|a| < 1$

y_t converges to y* and the solution is **stable**. There are two subcases:

$0 < a < 1$,

Monotonic convergence.

$-1 < a < 0$

Damped oscillations.

$|a| > 1$

Divergence:

$a > 1$

Explosion.

$a < -1$

Explosive oscillations.

To understand these features see Figure 8.1.

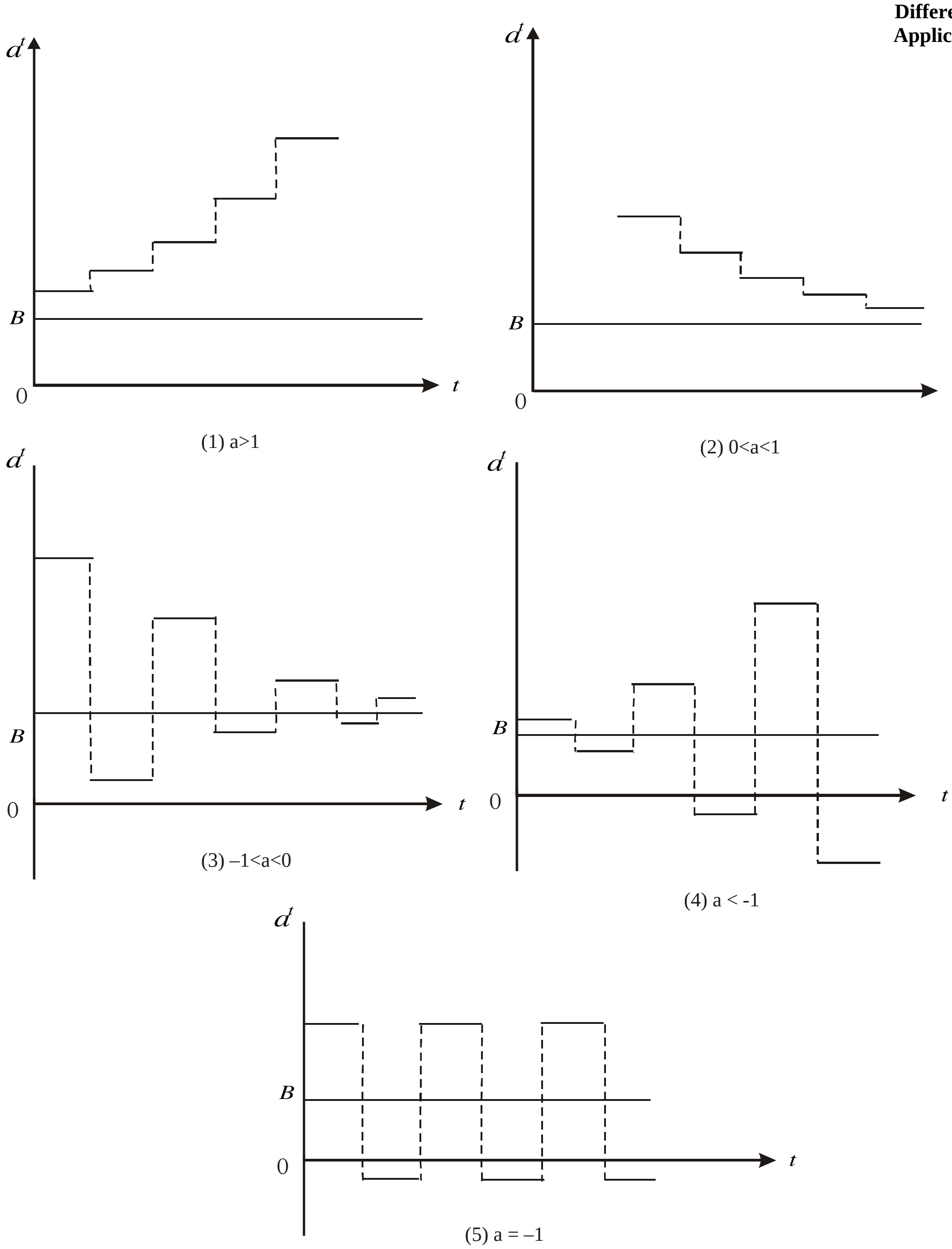

Fig. 8.1

In short,

$|a| > 1$ time path explodes (diverges)

$|a| < 1$ time path converges

$a > 0$ time path non-oscillating

$a < 0$ time path oscillating.

Thus, the condition for stability is $|a| < 1$.

The different cases are shown in Figure 8.1.

8.3.2 Economic Applications of First Order Equations

We consider three applications of the type of equations discussed in the previous section. The first is an analysis one sector Harrod-Domar model while the second is of price dynamics. The last one deals with the amortisation problem of hire purchase of consumer durables.

a) **Harrod-Domar One Sector Model**

An economy produces one good Q with capital K through a production function $Q_t = bK_t$, where b = constant productivity of capital. Accumulation of capital between t and t+1 is given by

$I_t = K_{t+1} - K_t$, where I_t = investment in t.

Saving $S_t = sQ_t$.

Equilibrium level of income is determined at the equality of savings and investment. So,

$S_t = I_t$

or, $sQ_t = K_{t+1} - K_t$.

Since $Q_t = bK_t$, we have $sbK_t = K_{t-1} - K_t$

or, $K_{t+1} = (1 + sb)K_t$, a homogeneous first order linear difference equation. Therefore, solution to this equation is given by

$K_t = (1 + sb)^t K_0$.

Since b is productivity of capital in the model, we write $\frac{1}{b}$ = capital output ratio = v (say).

Now $K_t = \left(1+\frac{s}{v}\right)^t K_0$ and

$Q_t = \left(1+\frac{s}{v}\right)^t Q_0$

Remember that $\frac{s}{v}$ = warranted rate of growth and constituted by two basic parameters s and v. We can find out the output growth rate given s and v.

b) **The Cobweb Model**

The essential feature of this model is that production or supply responds to price with a one-period lag. This type of lagged supply response is often observed for agricultural products.

We assume: 1) The market demand and supply functions are linear and do not change over time, 2) demand in any period t responds to price prevailing in the same period t, but supply in t depends on price that prevailed in the last period, (t – 1) and 3) the market is competitive in the sense that the price that prevails in each period is the price that equates demand and supply. Thus, the model can be set out as consisting of the following equations.

$D_t = a - b\, P_t$; a, b > 0

$S_t = \alpha - \beta\, P_{t-1}$; $\alpha, \beta > 0, \alpha < a$

$D_t = S_t$ for all t.

The first equation gives us the simple demand curve in period t. The second displays the lag in supply. Supply in t, S_t, is determined by prices of the immediately preceding period, P_{t-1}. The last equation is the condition of market clearing in each period. The three equations together yield a first order constant coefficient non-homogeneous difference equation in price.

$$P_t = \left(-\frac{\beta}{b}\right)P_{t-1} + \frac{a-\alpha}{b} \qquad \ldots (9)$$

With a, b α, β known, a specification of the initial price P_0 allows us to solve the equation as:

$$P_t = \left(P_0 - \frac{a-\alpha}{b+\beta}\right)\left(-\frac{\beta}{b}\right)^t + \frac{a-\alpha}{b+\beta} \qquad \ldots (10)$$

From our previous discussion, it is clear that the behaviour of P over time depends crucially on the term $\left(-\frac{\beta}{b}\right)$.

Since this term is negative (b, $\beta > 0$) the time path will always be oscillatory. Let us denote the constant $\frac{a-\alpha}{b+\beta}$ by P^*.

Then

$\left(\frac{\beta}{b}\right) > 1$ Price diverges

$\left(\frac{\beta}{b}\right) = 1$ Price oscillates uniformly

$\left(\frac{\beta}{b}\right) < 1$ Price converges to P^*.

Only in the last case (P_t approaches P^* as t increases), the system is **stable**. Thus, the condition for stability is $\left(\frac{\beta}{b}\right) < 1$. Since graphically $\left(\frac{1}{\beta}\right)$ is the slope of the supply curve and $\left(\frac{1}{b}\right)$ that of the demand curve in absolute value, the stability condition states that the slope of the supply curve must be steeper than the absolute value of the slope of the demand curve.

At this point, we pause to note the significance of the value $\frac{a-\alpha}{b+\beta}$. This is the constant value of price that is a solution of the equation (9). To check, substitute $P_t = P_{t-1} = P^*$ (a constant) in (9).

$$P^* = \left(-\frac{\beta}{b}\right)P^* + \frac{a-\alpha}{b}$$

or, $P^* = \frac{a-\alpha}{b+\beta}$.

Thus, $P_t = \frac{a-\alpha}{b+\beta}$ is a solution of (9). This type of constant solution is called **Stationary solution**. The price P^* may be called the **equilibrium price** because it equates demand and supply and stays unchanged over time.

Example: We want to investigate the behaviour of price in a market with the demand and supply functions:

$D_t = 86 - 0.8\ P_t$

$S_t = -10 + 0.2\ P_{t-1}$

Assuming market clearing in each period ($D_t = S_t$) we have

$(-0.8)\ P_t = 0.2\ P_{t-1} - 96$

or, $P_t = (-0.25)P_{t-1} + 120$

The solution is

$$P_t = \left(P_0 - \frac{120}{1+0.25}\right)(-0.25)^t + \frac{120}{1+0.25}$$

$= (P_0 - 96)\ (-0.25)^t + 96$

Since $|-0.25| = 0.25 < 1$, the time path of P is oscillating but converges. The market is stable and with the passage of time price approaches the equilibrium value 96.

c) **The Amortisation Problem**

We are all familiar with the practice of hire purchase or purchase by instalments of consumer durables like refrigerators, cars or T.V. sets. The buyer pays a part of the price at the time of purchase (the down payment) and pays the rest in monthly or annual instalments over a specified period. Because the payments are spread over a period of time, an interest cost is included in the value of instalments. **Amortisation** is the term associated with this method of repaying an initial debt plus interest charges by a series of payments of equal magnitude at equal intervals.

Let the value of the article purchased by V and P the down payment. Then the initial debt of the buyer is $D_0 = V - P$. The contract states that the debt, D_0 is to be paid off over T periods. The rate of interest is r (100 r%). The question we are interested in is: how is the magnitude of periodic instalment to be determined?

Let us denote the value of the instalment (still unknown) by B. This value stays constant over time. The outstanding debt D_t at the end of the period t obeys the equation

$D_t = (1 + r)D_{t-1} - B$ ………………………….(11)

This simply says that to find the outstanding debt at the end of the t^{th} period you take the debt outstanding at the end of the previous ($(t-1)^{th}$) period D_{t-1}, add the interest charge on it, rD_{t-1}, but subtract the payment B made in that period. Given the initial debt of D_0 the solution of (11) is

$$D_t = \left(D_0 + \frac{B}{r}\right)(1 + r)^t + \frac{B}{r} \quad \text{…. (12)}$$

The value of B is to be selected so that the debt disappears at the end of period T, that is, $D_t = 0$. From (12) we get

$$\left(D_0 + \frac{B}{r}\right)(1 + r)^t + \frac{B}{r} = 0$$

$$\text{or, } B = \frac{rD_0}{1-(1+r)^{-T}}$$

Thus, we have the exact relationship between the magnitude of the periodic payment and the rate of interest, the magnitude of the initial debt and the time horizon of the contract. The expression:

$$\left(\frac{1-(1+r)^{-T}}{r}\right)$$

is referred to as the **amortisation factor** and value of this factor has been extensively tabulated for different values of r and T.

Check Your Progress 1

1) What is a difference equation? Distinguish it from a differential equation.

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2) Discuss the nature of the following time paths

i) $y_t = 3^t + 1$ (ii) $y_t = 5\left(-\frac{1}{10}\right)^t + 3$

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3) Suppose you find the following the path of y.

$y_t = Aa^t + B$; $A < 0$, $B > 0$.

Draw the different cases of the behaviour of y_t for different values of a.

4) Solve the following equations:

i) $y_{t+1} - \frac{1}{3}y_t = 6$ for $y_0 = 1$

ii) $y_{t+1} - y_t = 3$ for $y_0 = 5$

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8.4 SOLVING SECOND ORDER DIFFERENCE EQUATIONS

A general **second-order difference equation** which we have already mentioned at begniging outset of this unit takes the form

$$y_{t+2} = f(t, y_t, y_{t+1}).$$

Just as in the case of first-order equation, a second-order equation will have a unique solution and can be derived by successive (recursive) calculation. We will show that given y_0 and y_1 there exists a uniquely determined value of y_t for all $t \geq 2$. Note that for a second-order equation we need two starting values, y_0 and y_1, in place of one taken in the first order counterpart.

8.4.1 Homogeneous Equations

Consider the following second order constant coefficient equation

$$y_{t+2} + ay_{t+1} + by_t = 0 \qquad \text{…. (13)}$$

We need to find two solutions of the equation above.

If we make a guess that the solution takes the form $u_t = m^t$

In order for u_t to be a solution, we must have

$$m^t(m^2 + am + b) = 0$$

or, if $m \neq 0$,

$$m^2 + am + b = 0.$$

This is called the **characteristic (or auxiliary) equation** of the difference equation and its solutions are

$$-(1/2)a \pm \sqrt{((1/4)a^2 - b)}.$$

8.4.2 Behaviour of Solutions of Homogeneous Equations

Looking at the component $\sqrt{((1/4)a^2 - b)}$, we distinguish three cases:

i) Distinct real roots

If $a^2 > 4b$, the characteristic equation has distinct real roots, and the general solution of the homogeneous equation is

$$Am_1^t + Bm_2^t,$$

where m_1 and m_2 are the two roots.

ii) Repeated real root

If $a^2 = 4b$, then the characteristic equation has a single root, and the general solution of the homogeneous equation is

$$(A + Bt)m^t,$$

where $m = -(1/2)a$ is the root.

ii) Complex roots

If $a^2 < 4b$, then the characteristic equation has complex roots, and the general solution of the homogeneous equation is

$$Ar^t \cos(\theta t + \omega),$$

where A and ω are constants, $r = \sqrt{b}$, and $\cos\theta = -a/(2\sqrt{b})$, or, alternatively,

$C_1 r^t \cos(\theta t) + C_2 r^t \sin(\theta t)$,

where $C_1 = A\cos\omega$ and $C_2 = -A\sin\omega$ (using the formula that $\cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y)$).

When the characteristic equation has complex root, the solution **oscillates**. Ar^t is the **amplitude** (which depends on the initial conditions) at time t, and r is **growth factor**. $\theta/2\pi$ is the **frequency** of the oscillations and ω is the **phase** (which depends on the initial conditions).

If $|r| < 1$ then the oscillations are **damped**; if $|r| > 1$ then they are **explosive**.

Stability

We say that a system of differential equations is **stable** if its long-run behavior is not sensitive to the initial conditions.

Consider the second-order equation

$$y_{t+2} + ay_{t+1} + by_t = c_t.$$

Write the general solution as

$y_t = Au_t + Bv_t + u_t^*$,

where A and B are determined by the initial conditions.

This solution is **stable** if the first two terms approach 0 as $t \to \infty$, for all values of A and B. In this case, for *any* initial conditions, the solution of the equation approaches the particular solution u_t^*. If the first two terms approach zero for all A and B, then u_t and v_t must approach zero. You can take $A = 1$ and $B = 0$ to see that u_t approaches zero. On the other hand, take take $A = 0$ and $B = 1$ to see that v_t approaches 0. A necessary and sufficient condition for this to be so is that the moduli of the roots of the characteristic equation be both less than 1. Note that the modulus of a complex number $\alpha + \beta i$ is $+\sqrt{(\alpha^2 + \beta^2)}$, which is the absolute value of number if the number is real.

There are two cases:

- If the characteristic equation has complex roots then the modulus of each root is $\sqrt{b}$ (the roots are $\alpha \pm \beta i$, where $\alpha = -a/2$ and $\beta = \sqrt{(b - (1/4)a^2)}$). So for stability need $b < 1$.
- If the characteristic equation has real roots then the modulus of each root is its absolute value. So for stability we need the absolute values of each root to be less than 1, or $|-a/2 + \sqrt{(a^2/4 - b)}| < 1$ and $|-a/2 - \sqrt{(a^2/4 - b)}| < 1$.

8.4.3 Non-homogeneous Equations

To find the general solution of the original equation

$y_{t+2} + ay_{t+1} + by_t = c_t$

we need to find one of its solutions. Suppose that $b \neq 0$.

The form of a solution depends on c_t.

Suppose that $c_t = c$ for all t. Then $y_t = C$ is a solution if $C = c/(1 + a + b)$ and if $1 + a + b \neq 0$;

if $1 + a + b = 0$ then try $y_t = Ct$; if that does not yield a solution, we have to try $y_t = Ct^2$.

8.4.4 An Economic Application of Second Order Non-homogeneous Equation

We discuss now an economic example of a second order non-homogeneous equation. This is Samuelson's model of interaction between the multiplier and the accelerator. Consider the following macro-economic equations:

$C_t = C_0 + cY_{t-1}$, $0 < c < 1$.

$I_t = I_0 + v(C_t - C_{t-1})$; $v > 0$.

$Y_t = C_t + I_t$

The symbols Y, C, I stand for national income, consumption and investment respectively. The first equation is the consumption function with a one-period lag; the second is the investment function of the accelerator type. $C_0 + I_0$ are the levels of autonomous consumption and investment. The marginal propensity to consume c and the accelerator coefficient v are assumed to be constant. The final equation is the condition of macro balance. The three equations together generate the following difference equation in Y

$$Y_t - c(1 + v)\, Y_{t-1} + cvY_{t-2} = C_0 + I_0 \qquad \dots (15)$$

The characteristic equation for the homogeneous part is

$$m^2 - c(1 + v)\, m + cv = 0$$

The roots are

$$m_1, m_2 = \frac{1}{2}\left(c(1 + v) \pm \sqrt{c^2(1+v)^2 - 4cv}\right) \qquad \dots (16)$$

Both m_1 and m_2 are positive because from the theory of quadratic equations we know $m_1 + m_2 = c(1 + v) > 0$ and $m_1m_2 = cv > 0$. Since $c(1 + v) - cv \neq 1$, the **particular** solution is $\frac{C_0 + I_0}{1-c}$. Three types of solution are possible depending on the values of c and v.

1) $c^2(1 + v)^2 > 4cv$

   or, $c(1+v)^2 > 4v$, the roots are real and distinct.

   Here,

   $$Y_t = A_1 m_1^t + A_2 m_2^t + \frac{C_0 + I_0}{1-c}; \; A_1, A_2 \neq 0 \text{ and constants}$$

2) $c(1+v)^2 = 4v$, the roots are real and equal with value $\frac{1}{2}c(1+v)$. In this case

   $$Y_t = (A_1 + A_2t)\left(\frac{c(1+v)^2}{2}\right) + \frac{C_0 + I_0}{1-c};$$

3) $c(1+v)^2 < 4v$, the roots are complex. From (16) we see that the roots are of the form $(a \pm ib)$ with

   $$a = \frac{1}{2}c(1+v)$$

$b = \frac{1}{2}\sqrt{4cv - c^2(1+v)^2}$

$(\Theta\ \sqrt{c^2(1+v)^2 - 4cv} = \sqrt{i^2\left(4cv - c^2(1+v)^2\right)} = i\sqrt{4cv - c^2(1+v)^2} = ib)$

The modulus of the roots

$r = \sqrt{a^2 + b^2} = \sqrt{cv}$

The solution is

$Y_t = (\sqrt{cv})^2\ (A_1 \cos(t\theta) + A_2 \sin(t\theta)) + \frac{C_0 + I_0}{1-c}$

where $\theta = \tan^{-1}\left(\frac{\sqrt{4cv - c^2(1+v)^2}}{c(1+v)}\right)$

In this case, we have a **cyclical** time path of national income Y. If $\sqrt{cv} < 1$, then $(\sqrt{cv})^t$ will tend to zero as t increases and Y_t will approach the value $\frac{C_0 + I_0}{1-c}$.

Thus, the condition for stability (damped oscillations in Y) is $\sqrt{cv} < 1$, that is, the product of the marginal propensity to consume and the accelerator coefficient should be less than unity.

Check Your Progress 2

1) Solve the following difference equations and determine whether the solution paths are convergent or divergent, oscillating or not.

   a) $y_{t+2} + 3y_{t+1} - (7/4)y_t = 9$.

   b) $y_{t+2} - 2y_{t+1} + 2y_t = 1$.

   c) $y_{t+2} - y_{t+1} + (1/4)y_t = 2$.

   d) $y_{t+2} + 2y_{t+1} + y_t = 9 \cdot 2^t$.

   e) $y_{t+2} - 3y_{t+1} + 2y_t = 3 \cdot 5^t + \sin((1/2)\pi t)$.

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2) Find the roots of the equation

   $y_t = ay_{t-1} + by_{t-2}$

Examine when the roots are

1) real, unequal

2) real, equal

3) complex

   What is the auxiliary or the characteristic equation of the equation above? What are the final forms of general solution of the equation in each case?

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3) Find the solutions of the equations:

a) $y_t + 4y_{t-2} = 0$, y = 12, 11 at t = 0, 1 respectively.

b) $y_t = 2y_{t-2} - \frac{4}{3}y_{t-2}$, y = 0, 1 at t = 0, 1 respectively.

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8.5 LET US SUM UP

In continuation with the theme on solving economic problems in a dynamic set up, the present unit took up 'time' as a discrete independent variable and examined tool of simple difference equations. In the process, we considered the solutions of first and second order linear difference equations covering homogeneous and non-homogeneous cases. To see the applications of these equations to economic problems, the time path of adjustment of macro-economic variable – national income – in case of the simple Keynesian multiplier model and Samuelson's multiplier-accelerator interaction model were discussed. We also examined the time path of adjustment of the price variable and looked into the conditions of dynamic stability of the different systems – i.e., whether over time, the economic variables – price or national income – converge to a stable equilibrium. Finally, the conditions when the systems become dynamically explosive – i.e., the variables move further and further away from the equilibrium value were examined.

8.6 KEY WORDS

Amortisation: It is the term associated with the method of repaying an initial debt plus interest charges by a series of payments of equal magnitude at equal intervals.

Cobweb Model: A model where production or supply responds to price with a one-period lag. This model is often used to analyse the demand-supply mechanism for markets of agricultural commodities.

Constant Coefficient Difference Equation: A difference equation has constant coefficient if the coefficients a_i's associated with the y values are constant and do not change over time.

Difference Equation: A difference equation is an equation involving the values of an unknown function y(x) for different values of x. The independent variable – time in problems of economic dynamics – takes only discrete values. The form of the equation is, $y_t = a_1y_{t-1} + a_2y_{t-2} + \ldots + a_ny_{n-1} + b$, where

a_1, a_2, …., a_n and b are constants, is an example of an n-th order linear, constant coefficient, difference equation.

Homogeneous Difference Equation: A difference equation is homogeneous if the constant term b is zero.

Linear Difference Equation: A difference equation is linear if (i) the dependent variable y is not raised to any power and (ii) there are no product terms.

Non-homogeneous Difference Equation: A difference equation is non-homogeneous if the constant term, b, is non-zero.

Order of a Difference Equation: It is determined by the maximum number of periods lagged.

8.7 SOME USEFUL BOOKS

Allen, R.G.D., 1959, *Mathematical Economics* (Second Edition) St. Martin's Press Inc., New York.

Baumol, W.J., 1974, *Economic Dynamics* (Second Edition) Macmillan, New York. Chapters 9, 10 and 11.

Chiang, Alpha C. 1984, *Fundamental Methods of Mathematical Economics* (Third Edition): Mc-Graw Hill International Edition, New Delhi.

8.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 8.1

2) i) Non-oscillatory; divergent

   ii) Oscillatory; convergent

3) Fig. 8.1 shows the time path for A > 0. Draw the corresponding figures for A < 0.

4) i) $y_t = -8\left(\frac{1}{3}\right)^t + 9$; ii) $y_t = -2\left(\frac{-1}{4}\right)^t + 4$

Check Your Progress 2

1)

   a) $A_1\left(\frac{1}{2}\right)^t + A_2\left(\frac{-7}{2}\right)^t + 4$. Nonconvergent oscillations.

   b) $\left(\sqrt{2}\right)^t\left(A_1\cos\left(\frac{\pi}{4}\right)t + A_2\sin\left(\frac{\pi}{4}\right)t\right) + 1$. Nonconvergent oscillation.

   c) $A_1\left(\frac{1}{2}\right)^t + A_2 t\left(\frac{1}{2}\right)^t + 8.$ Convergent, non-oscillating.

   d) The characteristic equation is $m^2 + 2m + 1 = (m+1)^2 = 0$, which has a double root of –1. So the general solution of the homogeneous equation is $y_t = \left(C_1 + C_2 t\right)(-1)^t$. A particular solution is obtained by inserting $u_t^* = A2^t$, which yields $A = 1$. So the general solution of the inhomogeneous equation is $y_t = (C_1 + C_2 t)(-1)^t + 2^t$.

e) By using the method of undetermined coefficients the constants A, B, and C in the particular solution $u^* = A5^t + B\cos\left(\frac{\pi}{2}\right)t + C\sin\left(\frac{\pi}{2}\right)t$, we obtain $A = 1/4$, $B = 3/10$, and $C = 1/10$. So the general solution to the equation is

$$y_t = C_1 + C_2 2^t + \left(\frac{1}{4}\right)5^t + \left(\frac{3}{10}\cos\left(\frac{\pi}{2}\right)t\right) + \left(\frac{1}{10}\sin\left(\frac{\pi}{2}\right)t\right).$$

2) See Section 8.4 and answer.

Note that in the text $y_t = ay_{t-1} + by_{t-1}$

Here you have a slightly changed equation.

3) a) $y_t = 2^t\left[12\cos\left(\frac{t\pi}{2}\right) + \frac{11}{2}\sin\left(\frac{t\pi}{4}\right)\right]$.

b) $y_t = \left(\frac{2}{\sqrt{3}}\right)^t\left(\frac{\sqrt{3}}{2\sin\theta}\right)\sin(t\theta)$

where $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

8.9 EXERCISES

1) Investigate the behaviour of price in a market, i.e., the stability of a system with demand and supply functions:

a) $D_t = 86 - 0.8\,P_t$

$S_t = -10 + 0.8\,P_{t-1}$

b) $D_t = 86 - 0.8\,P_t$

2) What is amortisation? Derive the exact relationship between the magnitude of the periodic payment B and the rate of interest r, the magnitude of the initial debt D_0 and the time horizon of the contract T.

3) Establish the stability condition of Samuelson's multiplier-accelerator interaction model.

4) Find the time path represented by the equation $y_t = 2\left(-\frac{4}{5}\right)^t + 9$.

5) Find the solution of the equation $y_{t+1} + \frac{1}{4}y_t = 5$ for $y_0 = 2$.

5) The demand and supply for cobweb model is given as $Q_{dt} = 19 - 6P_t$ and $Q_{st} = 6P_{t-1} - 5$. Find the intertemporal equilibrium price and comment on the stability of the equilibrium.

Answer or Hints to Exercises

1) Note that for the Cobweb model

$D_t = a - b\,P_t \qquad a, b > 0$

$S_t = \alpha + \beta P_{t-1}\ \alpha, \beta > 0, \alpha < 0$

$D_t = S_t$ for all t.

A specification of the initial price P_0 allows us to solve the equation.

$$P_t = \left(-\frac{\beta}{b}\right)P_{t-1} + \frac{a-\alpha}{b}$$

$$\text{as } P_t = \left(P_0 - \frac{a-\alpha}{b+\beta}\right)\left(-\frac{\beta}{b}\right)^t + \frac{a-\alpha}{b+\beta}$$

$$\text{or, } P_t = \left(P_0 - P^*\right)\left(-\frac{\beta}{b}\right)^t + P^*; P^* = \frac{a-\alpha}{b+\beta}$$

If $\left(-\frac{\beta}{b}\right)^t \rightarrow 0$ as $t \rightarrow \infty$, $P_t \rightarrow P^*$, the equilibrium value.

This is possible if and only if $\frac{\beta}{b} < 1$

i.e., $\beta < b$.

a) $\beta = 0.8$, $b = 0.8$. Hence $\beta = b$.

This results in uniform oscillation, as $\frac{1}{\beta} = \frac{1}{b}$.

or, the slope of the supply curve = the absolute slope of the demand curve.

b) $b = 0.8$, $\beta = 0.9$

Hence $\frac{\beta}{b} = 1$

or, $\beta > b$.

or, $\frac{1}{\beta} < \frac{1}{b}$.

i.e., the slope of the supply curve is less than the absolute value of the slope of the demand curve.

Hence, price diverges further and further away from the equilibrium and you come across an explosive and oscillatory situation.

c) $B = 0.9$, $\beta = 0.8$

$\frac{\beta}{b} < 1$ implies damped oscillation and the system is stable.

You should draw diagrams in each case and satisfy yourself.

2) See Example (b) in Sub-section 8.3.4.

3) Sub-section 8.4.4 and answer.

4) Since $b = -\frac{4}{5} < 0$, the time path is oscillatory. As $|b| = \frac{4}{5} < 1$, the oscillation is damped and it converges to equilibrium level of 9.

5) $y_t = -2\left(-\tfrac{1}{4}\right)^t + 4.$

6) $\bar{p} = 2$; discuss on uniform oscillation.